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Discontinuous Solutions in Space Problems of the Calculus of Variations.

By Lawrence M. Graves.

In his dissertation * Caratheodory treated the extension of the Jacobi condition to minimizing curves with corners, the so-called discontinuous solutions. Bolza † and Dresden ‡ have completed the theory in certain important respects. The form of statement of the Caratheodory condition and the method of proof heretofore given seem to be not easily extensible to space problems. The proof assumes the existence of the points conjugate to the corner on both extremal arcs meeting there, and it also involves rather complicated computations. Moreover, the condition has not been given a readily applicable analytic form.

I shall give two (equivalent) new formulations of the Jacobi-Caratheodory condition, applicable to minimizing curves in any number of dimensions and having any number of corners. The first is in terms of the family of extremaloids or broken extremals through the first end-point of the minimizing curve. A natural method of proof is by means of the envelope theorem and by a simple geometric construction at the corners. This method requires the usual additional assumptions and does not apply on both sides of corners. The second formulation makes use of the second variation and a new definition of conjugate point in terms of solutions of the (generalized) Jacobi equations for the discontinuous problem. The proof requires no special assumptions, other than that the function Ω_0 does not vanish at the corners, and leaves no exceptions uncared for. It is shown moreover that no minimizing curve can have a corner at which $\Omega_0 \neq 0$ and the two branches meet in a cusp. Neither of the new methods requires the assumption of the existence of the points conjugate to the corners on the various extremal arcs. Finally, I give a rigorous analytic proof of sufficient conditions for a minimum.

---

* *Über die diskontinuierlichen Lösungen in der Variationsrechnung.* (Göttingen, 1904). See also "Über die starken Maxima und Minima bei einfachen Integralen," *Mathematische Annalen*, Vol. 62 (1906), p. 474.

† *American Journal of Mathematics*, Vol. 30 (1908), p. 209; *Vorlesungen über Variationsrechnung*, Ch. VIII.

‡ *Transactions of the American Mathematical Society*, Vol. 9 (1908), p. 480.

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1. *Statement of the problem.* We are to consider the problem of minimizing an integral of the form

$$J = \int_{t_0}^{T} F(x_1, \cdots, x_n, x_1', \cdots, x_n')\, dt = \int_{t_0}^{T} F(x, x')\, dt$$

in the class of all admissible curves

$$x^\alpha = x^\alpha(t) \qquad (t_0 \leqq t \leqq T;\ \alpha = 1, \cdots, n)$$

which join two fixed points $x_0{}^\alpha$ and X^α in space of n dimensions $(n > 1)$. We assume the usual continuity and homogeneity properties for the integrand function F, and "admissible curves" are ordinary curves in the sense used by Bolza.* We shall make use of the well-known summation convention of

* *Vorlesungen*, § 25, p. 189.

tensor analysis, except that it will not be necessary to distinguish between contravariant and covariant indices. We have, for example, to deal with the "norm" $x'^a x'^a$ $(= \sum_{a=1}^{n} x'^a x'^a)$ of a vector x'^a. It is important to recall that if the x'^a are the derivatives of functions x^a defining an admissible curve, then $x'^a x'^a$ never vanishes along that curve. We shall use the indices α and β to run from 1 to n, and the indices i and j to run from 1 to $n-1$.

2. *The classical necessary conditions for a minimum.* An admissible curve

$$E_0 : x^a = x^a(t)$$

minimizing the integral J must satisfy the following well-known necessary conditions:

(I_1) the *Euler equations:*

$$\frac{d}{dt}\frac{\partial F}{\partial x'^a} = \frac{\partial F}{\partial x^a} \tag{2.1}$$

between corners on E_0;

(I_2) the *corner conditions:*

$$\frac{\partial F(x, x'^-)}{\partial x'^a} = \frac{\partial F(x, x'^+)}{\partial x'^a} \tag{2.2}$$

at each corner; here x'^- and x'^+ denote the directions on E_0 preceding and following the corner, respectively;

(II) the *Weierstrass condition:*

$$\mathcal{E}(x, x', \bar{x}') \equiv \left[\frac{\partial F(x, \bar{x}')}{\partial x'^a} - \frac{\partial F(x, x')}{\partial x'^a}\right]\bar{x}'^a \geqq 0$$

for every (x, x') on E_0 and every direction $\bar{x}'$ different from x';

(III) the *Legendre condition:*

$$Q(x, x'; \eta) \equiv \eta^\alpha \frac{\partial^2 F(x, x')}{\partial x'^\alpha \partial x'^\beta} \eta^\beta \geqq 0$$

for every (x, x') on E_0 and every η whose components are not proportional to those of x';

(III_0) the function F_1 defined by the equations *

---

* Cf. Bliss, *Transactions of the American Mathematical Society*, Vol. 15 (1914), pp. 376, 378.

$$F_1 x'^\alpha x'^\beta = H^{\alpha\beta}$$

is not negative along E_0; here the matrix $H^{\alpha\beta}$ is the adjoint of $\partial^2 F/\partial x'^\alpha \partial x'^\beta$;

(IV) the *Jacobi condition:* that there shall be no pair of conjugate points on any arc of E_0 which is of class C' and has $F_1 \neq 0$ along it.

Geometrically interpreted, the Jacobi condition is that no arc of E_0 of class C' may contain a point of contact with the envelope, if existing, of the $(n-1)$-parameter family of extremal arcs passing through a fixed point of the arc of E_0 in question. If the equations of this family of extremal arcs are

$$x^\alpha = \psi^\alpha(t, a)$$

then the analytic formulation of the condition is that the determinant

$$\begin{vmatrix} \partial\psi^\alpha/\partial a^\iota & \partial\psi^\alpha/\partial t \end{vmatrix}$$

shall not vanish along the arc of E_0 in question, except at the fixed point through which all the extremals of the family pass. A formulation of the Jacobi condition for minimizing arcs of class C', in terms of the second variation and solutions of the Jacobi equations, has been given by Bliss.* A modification of Bliss' method is used in §§ 8-10 of this paper.

3. *Construction of families of extremaloids.* The equations (2.1) and (2.2) characterize the admissible curves called *simple extremaloids* by Tonelli.† We shall drop the prefix "simple," inasmuch as no other types of extremaloids appear in this paper.

If E_0 is an extremal arc

$$x^\alpha = x^\alpha(t) \qquad (t_1 \leqq t \leqq t_2)$$

along which the function F_1 does not vanish, then for each point x_0 and direction x_0' sufficiently near those on E_0 there is a unique extremal passing through x_0 in the direction x_0'. The equations of these extremals can be given in the form

$$x^\alpha = \Xi^\alpha(t, x_0, x_0')$$

where the functions Ξ^α and their first and second derivatives with respect to t are of class C' for the specified range of x_0 and x_0' and for t on an interval $t_1 - \delta \leqq t \leqq t_2 + \delta$, and where

$$\Xi^\alpha(t, x_0, x_0') = x^\alpha(t) \qquad (t_1 \leqq t \leqq t_2)$$

---

* "Jacobi's Condition for Problems of the Calculus of Variations in Parametric Form," *Transactions of the American Mathematical Society*, Vol. 17 (1916), p. 195.

† *Fondamenti di Calcolo delle Variazioni*, Vol. II, p. 189.

whenever x_0 and x_0' represent a point and corresponding direction on the original extremal E_0. The totality of these extremals constitutes a $(2n-2)$-parameter family

$$x^a = \Psi^a(t, a, b)$$

where a and b each represent $n-1$ parameters. The totality of extremals through a fixed point constitutes an $(n-1)$-parameter family

$$x^a = \psi^a(t, a).$$

The functions Ψ^a and ψ^a have the same differentiability properties as Ξ^a. Moreover the determinant

$$(3.1) \qquad \begin{vmatrix} \frac{\partial \Psi^a}{\partial a^i} & \frac{\partial \Psi^a}{\partial b^i} & \frac{\partial \Psi^a}{\partial t} & 0 \\ \frac{\partial^2 \Psi^a}{\partial t \partial a^i} & \frac{\partial^2 \Psi^a}{\partial t \partial b^i} & \frac{\partial^2 \Psi^a}{\partial t^2} & \frac{\partial \Psi^a}{\partial t} \end{vmatrix}$$

never vanishes for $t_1 - \delta \leqq t \leqq t_2 + \delta$, and the determinant

$$(3.2) \qquad \begin{vmatrix} \frac{\partial \psi^a}{\partial a^i} & \frac{\partial \psi^a}{\partial t} \end{vmatrix}$$

is not identically zero on any subinterval of $t_1 t_2$.*

We shall obtain corresponding imbedding theorems for extremaloids, by starting from a family of extremals containing an extremal arc of the given extremaloid E_0, and showing how to get past a corner.† The extension of the properties of the determinants (3.1) and (3.2) will come naturally in the later sections on the second variation and the Jacobi equations.

Suppose then that E_0 is an extremaloid joining the points x_0 and X and having corners at $x_1, \cdots, x_m$, and suppose that the function F_1 is different from zero along E_0, including both sides of corners. For definiteness and simplicity of notation, start from a family $x^a = \psi^a(t, a)$ containing the first extremal arc E_{01} of E_0 for $a^i = a_0{}^i$, $t_0 \leqq t \leqq t_1$. Let p^{a-} and p^{a+} repre-

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* These statements are justified on the basis of the imbedding theorems for differential equations. Cf. Bolza, *Vorlesungen über Variationsrechnung*, Ch. IV; Bliss and Mason, *Transactions of the American Mathematical Society*, Vol. 9 (1908), pp. 443, 444; J. H. Taylor, *Bulletin of the American Mathematical Society*, Vol. 31 (1925), p. 257.

† Cf. the references to Caratheodory and Bolza already cited; also Sakellariou, "Sur les solutions discontinues du problème du calcul des variations dans l'espace à n dimensions," *Comptes Rendus du Congrés Internationale des Mathématiciens*, Strasbourg (1924), p. 351.

sent the direction cosines of the tangents to E_0 at the corner x_1, and suppose for convenience that $\psi^{a\prime}(t_1, a_0) = p^{a-}$.* The corner equations

$$(3.3) \qquad \frac{\partial F[\psi(t,a),q]}{\partial x'^a} - \frac{\partial F[\psi(t,a),\psi'(t,a)]}{\partial x'^a} = 0, \qquad q^a q^a - 1 = 0$$

have the initial solution $a^i = a_0{}^i$, $t = t_1$, $q^a = p^{a+}$. At this solution the functional determinant with respect to the $n+1$ variables (t, q) is

$$(3.4) \qquad \begin{vmatrix} \dfrac{\partial^2 F^+}{\partial x'^a \partial x'^\beta} & \dfrac{\partial^2 F^+}{\partial x'^a \partial x^\beta}\, p^{\beta-} - \dfrac{\partial F^-}{\partial x^a} \\ 2p^{\beta+} & 0 \end{vmatrix},$$

since the arc E_{01} satisfies the Euler equations (2.1). If we expand this determinant by the last row and column and make use of the identities †

$$p^{a+} F_1{}^+ p^{\beta+} = \text{adjoint of } \frac{\partial^2 F^+}{\partial x'^a \partial x'^\beta},$$

$$p^{a+} \frac{\partial^2 F^+}{\partial x'^a \partial x^\beta} = \frac{\partial F^+}{\partial x^\beta},$$

we find that the determinant (3.4) reduces to $2F_1{}^+ \Omega_0$, where

$$\Omega_0(x, p^-, p^+) = (\partial F^+/\partial x^a) p^{a-} - (\partial F^-/\partial x^a) p^{a+}.$$

Consequently, *if we make the additional assumption*

$$\Omega_0 \neq 0,$$

the equations (3.3) have unique solutions $t = t(a)$, $q^a = q^a(a)$, having a, t, q in sufficiently restricted neighborhoods of a_0, t_1, p^+ respectively, and these solutions are of class C' at least. Application of the imbedding theorems to the external arc E_{02} adjacent to E_{01} shows that through each point $x^a = \psi^a(t(a), a)$ there passes a unique extremal arc in the direction $x'^a = q^a(a)$, provided a^i is sufficiently near $a_0{}^i$.

If the function Ω_0 is different from zero at each of the corners on E_0, we obtain by this method (with proper choice of the parameter t on each sub-arc) *a family $x^a = \phi^a(t, a)$ of extremaloids, defined for $t_0 - \delta \leqq t \leqq T + \delta$, containing the original extremaloid E_0 for $a = a_0$, $t_0 \leqq t \leqq T$. The functions ϕ^a are continuous in all their arguments, and ϕ^a, $\partial\phi^a/\partial t$, and $\partial^2\phi^a/\partial t^2$ are of class C' in all their arguments except at the corners.* There is no change in the reasoning if we start from a family of extremals depending on a different number of parameters. We shall use the notation

---

* Primes will always denote differentiation with respect to t.

† These identities are consequences of the homogeneity condition on the integrand function F.

$$x^a = \Phi^a(t, a, b)$$

to represent the general $(2n-2)$-parameter family of extremaloids, containing E_0 for $a^i = a_0{}^i$, $b^i = b_0{}^i$. That we obtain the same set of curves, no matter which extremal arc of E_0 we start from, is a consequence of the fact, to be proved in § 10 for the case of extremaloids, that if the determinant

$$\begin{vmatrix} \frac{\partial \Phi^a}{\partial a^i} & \frac{\partial \Phi^a}{\partial b^i} & \frac{\partial \Phi^a}{\partial t} & 0 \\ \frac{\partial^2 \Phi^a}{\partial t \partial a^i} & \frac{\partial^2 \Phi^a}{\partial t \partial b^i} & \frac{\partial^2 \Phi^a}{\partial t^2} & \frac{\partial \Phi^a}{\partial t} \end{vmatrix}$$

is different from zero at one point of E_0 then it never vanishes.

It is sometimes convenient to have the corners on all the extremaloids of a family occur for the same values of the parameter t. This can be brought about by a transformation of the form

$$t = f(\bar{t}, a, b),$$

where f is continuous in all its arguments, f and $f' = \partial f / \partial \bar{t}$ are of class C' between corners, and $f' > 0$. Such a transformation multiplies the determinants

$$\begin{vmatrix} \frac{\partial \phi^a}{\partial a^i} & \frac{\partial \phi^a}{\partial t} \end{vmatrix}, \quad \begin{vmatrix} \frac{\partial \Phi^a(t)}{\partial a^i} & \frac{\partial \Phi^a(t)}{\partial b^i} & \frac{\partial \Phi^a(t)}{\partial t} & 0 \\ \frac{\partial \Phi^a(t_0)}{\partial a^i} & \frac{\partial \Phi^a(t_0)}{\partial b^i} & 0 & \frac{\partial \Phi^a(t_0)}{\partial t} \end{vmatrix},$$

by $f'(\bar{t})$ and $f'(\bar{t})f'(\bar{t}_0)$ respectively, and hence cannot affect the sign of either of these determinants. When the corners on all the extremaloids of the family occur for the same values of the parameter t, the partial derivatives of the functions Φ^a (or ϕ^a) with respect to the parameters a^i and b^i are continuous functions of t.

4. *A relation between the functions $\mathcal{E}$ and Ω_0, and a further necessary condition.* The relation sought is due to Dresden.* We temporarily select the arc length s as parameter along the extremaloid E_0. Then we find preceding a corner

$$d/ds\, \mathcal{E}\,[x(s), x'(s), p^+] = \Omega_0,$$

and following a corner

$$d/ds\, \mathcal{E}\,[x(s), x'(s), p^-] = -\Omega_0,$$

* *Transactions of the American Mathematical Society*, Vol. 9 (1908), p. 485.

where p^{a-} are the direction cosines of the tangent to E_0 preceding the corner, and p^{a+} those of the tangent following the corner. From the corner conditions (2.2) we obtain the equation

$$\mathcal{E}(x, p^-, p^+) = 0 = \mathcal{E}(x, p^+, p^-).$$

From this and the Weierstrass condition we readily obtain the following theorems.

If E_0 is an extremaloid along which $\mathcal{E} \geqq 0$, then $\Omega_0 \leqq 0$ at the corners.

If E_0 is an admissible arc minimizing the integral J, then $\Omega_0 \leqq 0$ at the corners on E_0.

5. *Additional properties of extremals and extremaloids.* The Legendre quadratic form $Q(x, x'; \eta)$ is said to be *positive regular* in case it is never negative and vanishes only when the η^a are proportional to x'^a. The form Q can be transformed by a real orthogonal transformation into a form $\bar{Q} = \lambda_a \zeta^a \zeta^a$ whose coefficients λ_a are the n roots of the characteristic equation of the matrix $\partial^2 F / \partial x'^a \partial x'^\beta$.* The characteristic equation in this case has the form when expanded

$$(-1)^n \lambda^n + \cdots\cdots - F_1 x'^a x'^a \lambda = 0.$$

Consequently we have

LEMMA 5.1. *If the form Q is never negative and $F_1 > 0$, then Q is positive regular, and conversely, if Q is positive regular, then $F_1 > 0$.*

Since the roots λ_a of the characteristic equation are continuous functions of (x, x'), we have the further property:

LEMMA 5.2. *If the Legendre quadratic form Q is positive regular for (x, x') in a bounded closed set S, then Q is positive regular in a neighborhood of S.*

The Weierstrassian function $\mathcal{E}(x, x', \bar{x}')$ obviously vanishes whenever x' and $\bar{x}'$ represent the same direction, and we wish to secure a related function which shall vanish only when the function $\mathcal{E}$ vanishes for x' and $\bar{x}'$ in different directions. This has been done by Bliss for space problems as follows.† It is first shown that for every two directions p and q there is at least one direction p_1 orthogonal to p such that

$$q^a = p^a \cos\omega + p_1{}^a \sin\omega \tag{5.1}$$

* Cf. Kowalewski, *Einführung in die Determinantentheorie* (1909), § 116; Dickson, *Modern Algebraic Theories*, p. 76, theorem 12.

† The "Weierstrass $\mathcal{E}$-Function for Problems of the Calculus of Variations in Space," *Transactions of the American Mathematical Society*, Vol. 15 (1914), p. 369.

where $0 \leqq \omega \leqq \pi$.* For every pair of orthogonal directions p and p_1 we define: a direction q depending also on ω by equation (5.1), and directions q_1 and q_2 depending also on τ by

$$q_1{}^a = p^a \cos\tau + p_1{}^a \sin\tau, \qquad q_2{}^a = -p^a \sin\tau + p_1{}^a \cos\tau, \qquad (0 \leqq \tau \leqq \omega).$$

Then we define the desired new function $\mathcal{E}_1$ by the equations

$$\mathcal{E}_1(x, p, p_1, \omega) = \mathcal{E}(x, p, q)/(1 - \cos\omega) \qquad (0 < \omega \leqq \pi),$$
$$\mathcal{E}_1(x, p, p_1, 0) = Q(x, p; p_1),$$

where Q is the Legendre quadratic form. It is easily shown that

$$\mathcal{E}_1(x, p, p_1, \omega) = Q(x, q_1{}^*; q_2{}^*) \tag{5.2}$$

where $q_1{}^*$ and $q_2{}^*$ correspond to a value $\tau = \tau^*$ between 0 and ω. Hence the function $\mathcal{E}_1$ is continuous in all its arguments when p_1 is orthogonal to p.†

We say that an extremal arc E is positively strong in case

$$\mathcal{E}_1(x, p, p_1, \omega) > 0$$

for every element (x, p) on E, p_1 orthogonal to p, and $0 \leqq \omega \leqq \pi$. Since such a set of points (x, p, p_1, ω) is closed, we have the property that:

LEMMA 5.3. *If an extremal arc E is positively strong, then there is a neighborhood of the elements (x, p) on E in which $\mathcal{E}_1(x, p, p_1, \omega) > 0$ for all directions p_1 orthogonal to p and $0 \leqq \omega \leqq \pi$.*

An additional proposition, obvious from the definition of the function $\mathcal{E}_1$, is the following:

LEMMA 5.4. *If $\mathcal{E}_1(x, p, p_1, \omega) > 0$ at an element (x, p) for all directions p_1 orthogonal to p and $0 \leqq \omega \leqq \pi$, then the quadratic form Q is positive regular at (x, p).*

An important theorem, due to Caratheodory for the plane case,‡ is as follows:

LEMMA 5.5. *If an extremal arc E ceases to be strong at a point x_1, but still has the quadratic form $Q(x_1, x_1'; \eta)$ regular, then there is an admissible*

---

* The components of p, q, etc., are supposed to represent direction cosines.

† The extra argument p_1 is essential because when $n > 2$ there is more than one direction orthogonal to p, and the limit of the ratio $\mathcal{E}(x, p, q)/(1 - \cos\omega)$ when q approaches p depends on the direction of approach.

‡ Cf. Bolza, *Variationsrechnung*, p. 387.

direction $\bar{p}$ *which with the direction* x_1' *of* E *at* x_1 *satisfies the corner conditions* (2.2).

To prove this, let $x^a = x^a(t)$ be equations of the arc E, and let $t = t_1$ correspond to the point x_1. Suppose for definiteness that E is positively strong for $t < t_1$. Let

$$\mathcal{E}(t, q) \equiv \mathcal{E}[x(t), x'(t), q], \; \mathcal{E}_1(t, p_1, \omega) \equiv \mathcal{E}_1[x(t), x'(t), p_1, \omega].$$

Then $\mathcal{E}_1(t, p_1, \omega) > 0$ for every $t < t_1$, p_1 orthogonal to $x'(t)$, and $0 \leqq \omega \leqq \pi$. Consequently $\mathcal{E}_1(t_1, p_1, \omega) \geqq 0$ for the same range of p_1 and ω. Since E ceases to be strong at x_1, there is a direction p_1 orthogonal to x_1', and ω such that $\mathcal{E}_1(t_1, p_1, \omega) = 0$. Moreover, $\omega \neq 0$, since Q is regular at (x_1, x_1'). Hence there is a direction $\bar{p}$ different from x_1', such that $\mathcal{E}(t_1, \bar{p}) = 0$. By Taylor's theorem we have

$$\mathcal{E}(t_1, q) = \mathcal{E}(t_1, \bar{p}) + \frac{\partial \mathcal{E}(t_1, \bar{p})}{\partial q^a}(q^a - \bar{p}^a) + R$$

$$= \left[\frac{\partial F(x_1, \bar{p})}{\partial x'} - \frac{\partial F(x_1, x_1')}{\partial x'^a}\right](q^a - \bar{p}^a) + R.$$

Since $\mathcal{E}(t_1, q) \geqq 0$ for every q, normed or not, and since R is infinitesimal of the second order with respect to the differences $q^a - \bar{p}^a$, we must have

$$(\partial/\partial x'^a) F(x_1, \bar{p}) = (\partial/\partial x'^a) F(x_1, x_1').$$

We shall say that an *extremaloid* E_0 is *positively strong* in case each of its extremal arcs is positively strong between corners, and the function F_1 remains positive on both sides of the corners on E_0. This requirement implies that the Legendre quadratic form Q is positive regular along E_0 including both sides of corners.

Consider a point x_1 and a direction p such that there is a unique direction p^+ different from p, satisfying the corner equations

$$(\partial/\partial x'^a) F(x_1, p) = (\partial/\partial x'^a) F(x_1, p^+).$$

In this case we shall say that at x_1 *there is a unique continuation direction* p^+ *corresponding to* p. It is obvious that p is then the unique continuation direction corresponding to p^+.

Suppose that E_0 is a positively strong extremaloid, having $\Omega_0 \neq 0$ at the corners, and with a unique continuation direction corresponding to each corner direction. By § 4, $\Omega_0 < 0$ at each corner, and if we continue past a corner on any extremal composing E_0, the function $\mathcal{E}(x, x', q)$ becomes negative for some directions q. If we require that the "strong" property is to

be preserved wherever possible, then an extremaloid having these properties may be said to be *uniquely determined by each of its elements* (x, x').

We can now state the following important theorem.

LEMMA 5.6. *Let E_0 be an extremaloid which is positively strong and which has $\Omega_0 \neq 0$ at each corner. Suppose also that the extremaloid E_0 is uniquely determined by each of its elements (x, x'). Let $x^a = \Phi^a(t, a, b)$ be equations of the $(2n-2)$ parameter family of extremaloids containing E_0 for $a^i = a_0{}^i$, $b^i = b_0{}^i$.*

Then there is a neighborhood of (a_0, b_0) in which every corresponding extremaloid of the family is positively strong.

The proof is rather complicated, and begins with the following preliminary proposition.

Suppose that E is an extremal arc with the properties:

1) *E has direction p at a point x_1, and at x_1 there is a unique continuation direction p^+ corresponding to p;*
2) *the quadratic form Q is positive regular at (x_1, p);*
3) *$F_1(x_1, p^+) \neq 0$;*
4) *$\Omega_0(x_1, p, p^+) \neq 0$;*
5) *the family $x^a = \Psi^a(t, a, b)$ of extremals contains E for $a^i = a_0{}^i$, $b^i = b_0{}^i$;*
6) *the parameter value t_1 corresponds to the point x_1 on E.*

Then there are neighborhoods $N(a_0, b_0)$ and $N(t_1)$ such that for each (a, b) in this $N(a_0, b_0)$ there is one and only one set of functions $[t(a, b), q^a(a, b)]$ having $t(a, b)$ in the $N(t_1)$ and the direction $q^a(a, b)$ different from $\Psi'^a[t(a, b), a, b]$, and satisfying the corner equations

$$\frac{\partial F[\Psi(t, a, b), q]}{\partial x'^a} - \frac{\partial F[\Psi(t, a, b), \Psi'(t, a, b)]}{\partial x'^a} = 0, \tag{5.3}$$

$$q^a q^a - 1 = 0.$$

For, suppose the last conclusion untrue. In constructing a family of extremaloids in § 3 we showed that there are always neighborhoods $N(a_0, b_0)$ and $N(t_1, p^+)$ in which there are unique functions $t(a, b)$, $q^a(a, b)$, satisfying the corner conditions. Then for every $N(a_0, b_0, t_1)$ there will be a point (a, b, t, q) such that:

1) (a, b, t) is in this $N(a_0, b_0, t_1)$;
2) (a, b, t, q) satisfies the corner equations (5.3);

3) the direction q is different from the direction $\Psi'(t, a, b)$;

4) (t, q) is not in the neighborhood $N(t_1, p^+)$.

The set composed of such points (a, b, t, q) has at least one limit point $(a_0, b_0, t_1, \bar{q})$. Since $\bar{q} \neq p^+$, we must have $\bar{q} = p$. Now at these points (a, b, t, q) we have

$$0 = \frac{\partial F(\Psi, q)}{\partial x'^a} - \frac{\partial F(\Psi, \Psi')}{\partial x'^a}$$

$$= \int_0^1 \frac{\partial^2 F[\Psi, \Psi' + u(q - \Psi')]}{\partial x'^a \partial x'^\beta} (q^\beta - \Psi'^\beta) du,$$

and therefore

$$0 = \int_0^1 Q[\Psi, \Psi' + u(q - \Psi'); q - \Psi'] du.$$

Since the quadratic form Q is positive regular at (x_1, p) it remains positive regular in a neighborhood of (x_1, p), and since the directions q and Ψ' are different we have

$$\int_0^1 Q[\Psi, \Psi' + u(q - \Psi'); q - \Psi'] du > 0$$

when (a, b, t, q) is sufficiently near (a_0, b_0, t_1, p). Thus the desired contradiction is secured.

To complete the proof of lemma 5. 6, suppose that the corners on E_0 correspond to parameter values $t_1, \cdots, t_m$. Then from the preliminary proposition we know that there exist neighborhoods

$$N(a_0, b_0), N(t_1), \cdots, N(t_m),$$

such that, for each (a, b) in $N(a_0, b_0)$ and each integer k there is only one parameter value $t_k(a, b)$ in $N(t_k)$ at which a corner is possible. Consider one of the extremal arcs E_{0k} of E_0. It has equations

$$x^a = \Phi^a(t, a_0, b_0) \qquad (t_k \leqq t \leqq t_{k+1}).$$

Let $\bar{t}_k > t_k$ be in $N(t_k)$, $\bar{t}_{k+1} < t_{k+1}$ be in $N(t_{k+1})$. Then by lemma 5. 3, there is a neighborhood $N_k(a_0, b_0)$ contained in $N(a_0, b_0)$ such that for (a, b) in $N_k(a_0, b_0)$ the arc

$$E_k : x^a = \Phi^a(t, a, b) \qquad (\bar{t}_k \leqq t \leqq \bar{t}_{k+1})$$

is positively strong. If there is a parameter value t in $N(t_k)$ at which the extremal containing the arc E_k ceases to be strong, there is a corner possible, by lemma 5. 5. Hence this $t = t_k(a, b)$. Thus we find that for each (a, b) in $N_k(a_0, b_0)$ the extremal arc

$$x^a = \Phi^a(t, a, b) \qquad [t_k(a, b) \leqq t \leqq t_{k+1}(a, b)]$$

is positively strong except at the ends. If the neighborhood $N_k(a_0, b_0)$ is taken sufficiently small, the function F_1 will remain positive at the ends of these extremal arcs. Since each extremaloid is composed of only a finite number of extremal arcs, we have proved the lemma.

GEOMETRIC TREATMENT OF THE JACOBI-CARATHEODORY CONDITION.

6. *The extension of the Jacobi condition. Let E_0 be an extremaloid joining the points x_0 and X and minimizing the integral J, and let*

$$x^a = \phi^a(t, a) \qquad (t_0 \leqq t \leqq T)$$

be equations of the $n-1$ parameter family of extremaloids passing through the point x_0 and containing E_0 for $a^i = a_0{}^i$. Then the determinant

$$D(t, a) = \begin{vmatrix} \dfrac{\partial \phi^a}{\partial a^i} & \dfrac{\partial \phi^a}{\partial t} \end{vmatrix}$$

*does not vanish for $a^i = a_0{}^i$, $t_0 < t \leqq T$, with the possible exception of the parameter values $t_k + 0$ corresponding to the sides of corners toward the second end-point X.**

The proof is made by means of the envelope theorem in the usual way, since (with the exception noted in the statement of the condition) there is a neighborhood of the point of contact with the envelope in which corners need not appear.† The differentiation of the integrals is slightly simplified if the corners occur for fixed values of t on all the extremaloids of the family.‡

The proof requires the following assumptions:

1) $F_1 \neq 0$ *along* E_0;
2) $\Omega_0 \neq 0$ *at the corners on* E_0;
3) *at the first zero of* $D(t, a_0)$ following t_0 *the partial derivative* $\partial D/\partial t$ *does not vanish;*
4) *the enveloping curve of the one parameter family of extremaloids containing* E_0 *and determined in the usual way, has a regressive branch at its point of contact with* E_0.

For the discussion of the existence and properties of the envelope, the

* This exception is removed in the proof by means of the second variation in §§ 8-10.

† For the proof cf. Bolza, *Variationsrechnung*, pp. 336, 378, 610; Bliss and Mason, *Transactions of the American Mathematical Society*, Vol. 9 (1908), pp. 449-451.

‡ See the closing paragraphs of § 3.

function F is supposed to be of class C^{iv} at least, so that all the operations of differentiation required in the theory may surely be carried out.

7. *A new form of the Caratheodory condition. Let E_0 be an extremaloid joining the points x_0 and X and minimizing the integral J, and let $\phi^a(t,a)$, $D(t,a)$ have the same meaning as in* § 6. *Then $D(t,a_0)$ does not change sign at the corners on E_0.*

The proof to be given in this section depends on the following assumptions:

1) *the extremaloid E_0 is positively strong;*
2) *$\Omega_0 \neq 0$ at the corners on E_0;*
3) *E_0 is uniquely determined by each of its elements (x, x').*

As indicated in the closing paragraphs of § 3, we may suppose in making the proof that the corners on all extremaloids of the family occur for fixed values of t. Then the partial derivatives $\partial\phi^a/\partial a^i$ are all continuous. Now suppose that the determinant D changes sign at a corner x_k where $t = t_k$. Let

$$x^a = \phi^{a-}(t,a) \qquad (t_0 \leqq t \leqq t_k + \delta)$$

denote the equations of the parts of the extremaloids preceding x_k, but with their last extremal arcs extended slightly beyond the corner manifold. Let

$$x^a = x^{a+}(t) \qquad (t_k - \delta \leqq t \leqq T)$$

denote the equations of the part of E_0 following x_k, but with its first extremal arc extended backward past the corner x_k. The equations

$$\phi^{a-}(t,a) = x^{a+}(u) \tag{7.1}$$

have the initial solution $t = t_k$, $a = a_0$, $u = t_k$, at which their functional determinant with respect to (a, t) is $D(t_k^-, a_0) \neq 0$. Hence these equations have a unique solution $a^i = \bar{a}^i(u)$, $t = \bar{t}(u)$ near this initial solution, and the functions $\bar{a}^i(u)$, $\bar{t}(u)$, are of class C'. Their derivatives $\bar{a}'^i$, $\bar{t}'$, satisfy the relations

$$(\partial\phi^{a-}/\partial t)\bar{t}' + (\partial\phi^{a-}/\partial a^i)\bar{a}'^i = x'^{a+}. \tag{7.2}$$

The one parameter family of curves E_u made up of the two parts

$$x^a = \phi^{a-}[t, \bar{a}(u)] \qquad [t_0 \leqq t \leqq \bar{t}(u)],$$
$$x^a = x^{a+}(t) \qquad (u \leqq t \leqq T),$$

are all continuous admissible curves joining the points x_0 and X. The value of the integral J taken along E_u is a function $J(u)$ which is differentiable. Its derivative turns out to be

$$J'(u) = \bar{a}'^i \int_{t_0}^{\bar{t}(u)} \left(\frac{\partial F}{\partial x^\alpha} \frac{\partial \phi^{\alpha-}}{\partial a^i} + \frac{\partial F}{\partial x'^\alpha} \frac{\partial^2 \phi^{\alpha-}}{\partial a^i \partial t} \right) dt + F^- \bar{t}' - F^+,$$

where the arguments of F^- are $\phi^{\alpha-}[\bar{t}(u), \bar{a}(u)]$, $\phi'^{\alpha-}[\bar{t}(u), \bar{a}(u)]$, and those of F^+ are $x^{\alpha+}(u)$, $x'^{\alpha+}(u)$. By means of integration by parts, the Euler equations (2.1) with the corner conditions (2.2), the homogeneity condition, and the relations (7.1) and (7.2), this expression reduces to

$$\begin{aligned} J'(u) &= \partial F^-/\partial x'^\alpha [(\partial \phi^{\alpha-}/\partial a^i)\bar{a}'^i + (\partial \phi^{\alpha-}/\partial t)\bar{t}'] - (\partial F^+/\partial x'^\alpha) x'^{\alpha+} \\ &= -x'^{\alpha+}(\partial F^+/\partial x'^\alpha - \partial F^-/\partial x'^\alpha) \\ &= -\mathcal{E}(\phi^{\alpha-}, \phi'^{\alpha-}, x'^{\alpha+}). \end{aligned}$$

If we solve equations (7.2) for the derivative $\bar{t}'$ at the value $u = t_k$, we find

$$\bar{t}'(t_k) = D(t_k + 0, a_0)/D(t_k - 0, a_0) < 0.$$

Consequently, for $u > t_k$ but near t_k we have $\bar{t}(u) < t_k$, and the point $\phi^{\alpha-}[\bar{t}(u), \bar{a}(u)]$ precedes the corner manifold on each extremaloid. Then by lemma 5.6 $\mathcal{E} > 0$ and hence $J'(u) < 0$ for u sufficiently near t_k, and hence E_0 could not minimize the integral J.

The Second Variation.*

8. *The extension of the Jacobi equations. Properties of their solutions.* If the extremaloid E_0 minimizes the integral J, then the second variation

$$J_2 \equiv \int_{t_0}^{T} 2\omega(t, \xi, \xi')\, dt$$

is greater than or equal to zero for all admissible variations $\xi^\alpha(t)$ such that $\xi^\alpha(t_0) = \xi^\alpha(T) = 0$. The quadratic form ω is defined by the equation

$$2\omega(t, \xi, \xi') = \xi^\alpha \frac{\partial^2 F}{\partial x^\alpha \partial x^\beta} \xi^\beta + 2\xi^\alpha \frac{\partial^2 F}{\partial x^\alpha \partial x'^\beta} \xi'^\beta + \xi'^\alpha \frac{\partial^2 F}{\partial x'^\alpha \partial x'^\beta} \xi'^\beta.$$

Here and in the remainder of the discussion of the second variation, the arguments of the derivatives of F are always the functions $x^\alpha(t)$, $x'^\alpha(t)$ defining the extremaloid E_0. Admissible variations $\xi^\alpha(t)$ are continuous on $(t_0 T)$ and have continuous first derivatives except at a finite number of points where one or more of the derivatives may have a finite jump. It will be con-

* The ensuing treatment of the second variation employs the elegant methods originated by Bliss. However, it is necessary to depart in a minor way from the methods for the parametric problem expounded in Bliss' paper, "Jacobi's Condition for Problems of the Calculus of Variations in Parametric Form," *Transactions of the American Mathematical Society*, Vol. 17 (1916), p. 195.

venient to say that such functions are of class D'. *It is assumed throughout that* $F_1 \neq 0$ *along the extremaloid* E_0, *and that* $\Omega_0 \neq 0$ *at the corners.*

If ξ is a variation giving to the second variation J_2 its minimum value zero, the functions ξ^a must satisfy the equations

$$\frac{\partial \omega}{\partial \xi'^a} = \int_{t_0}^{t} \frac{\partial \omega}{\partial \xi^a}\, dt + c_a \qquad (t_0 \leqq t \leqq T), \tag{8.1}$$

where the c_a are properly chosen constants. These equations are equivalent to the differential equations

$$\frac{d}{dt}\,\frac{\partial \omega}{\partial \xi'^a} = \frac{\partial \omega}{\partial \xi^a} \tag{8.2}$$

holding between corners of E_0 and ξ, and the corner conditions

$$\frac{\partial \omega(t^-, \xi, \xi'^-)}{\partial \xi'^a} = \frac{\partial \omega(t^+, \xi, \xi'^+)}{\partial \xi'^a}\,. \tag{8.3}$$

These equations may be obtained by applying to the integral J_2 the Euler equations (2.1) and the corner conditions (2.2), which are valid even if the integrand function 2ω is discontinuous in t.

The equations (8.2) are the Jacobi differential equations. They are not independent, but in fact are satisfied between corners on E_0 by every set of functions of the form $\rho x'^a$ where ρ is a differentiable function of t. The equations (8.1) are to be looked upon as the extension of the Jacobi equations for the case of discontinuous solutions. For these equations we have the following property.

LEMMA 8.1. *Every set of functions of the form* $\xi^a = \rho x'^a$ *is a solution of equations* (8.1), *provided* ρ *is a function of* t *of class* D' *on* $(t_0 T)$, *and* $\rho(t_1) = \cdots = \rho(t_m) = 0$.

The parameter values $t_1, \cdots, t_m$, as before, correspond to the corners on E_0. To prove the lemma, we note first that

$$\frac{\partial \omega(t, \rho x', \rho x'' + \rho' x')}{\partial \xi^{\beta'}} = \rho x'^a \frac{\partial^2 F}{\partial x^a \partial x'^\beta} + \rho x''^a \frac{\partial^2 F}{\partial x'^a \partial x'^\beta} + \rho' x'^a \frac{\partial^2 F}{\partial x'^a \partial x'^\beta} = \rho \frac{d}{dt}\,\frac{\partial F}{\partial x'^\beta} = \rho \frac{\partial F}{\partial x^\beta} \tag{8.4}$$

on account of the homogeneity property of F and the Euler equations (2.1). Similarly

$$\frac{\partial \omega(t, \rho x', \rho x'' + \rho' x')}{\partial \xi^\beta} = \rho x'^a \frac{\partial^2 F}{\partial x^a \partial x^\beta} + \rho x''^a \frac{\partial^2 F}{\partial x'^a \partial x^\beta} + \rho' x'^a \frac{\partial^2 F}{\partial x'^a \partial x^\beta} = \rho \frac{d}{dt}\,\frac{\partial F}{\partial x^\beta} + \rho' \frac{\partial F}{\partial x^\beta} = \frac{d}{dt}\left(\rho \frac{\partial F}{\partial x^\beta}\right).$$

Moreover, the corner conditions (8.3) are satisfied, since ρ is zero at $t_1, \cdots, t_m$, and ρ' disappears from the final form of $\partial\omega/\partial\xi'^\beta$ in (8.4).

We shall say that a solution ξ^a of equations (8.2) on an interval (t_{k-1}, t_k) is *normal* on that interval in case the functions ξ^a are of class C' and satisfy the equation $x'^a\xi'^a = 0$.* Using the assumption already made that $F_1 \neq 0$ along E_0, it is readily proved that for every normal solution the functions ξ^a are of *class* C'' at least. Solutions ξ^a normal on an interval (t_{k-1}, t_k) have the property that $\xi^a(\tau) = \xi'^a(\tau) = 0$ *at a point* τ *only if the functions* ξ^a *are identically zero on the interval.*† Moreover, if we start with an arbitrary solution ξ^a of equations (8.2) which is of class C' on (t_{k-1}, t_k), we can secure *an associated normal solution* $\xi_N{}^a$ of the form $\xi_N{}^a = \xi^a - \rho x'^a$, where ρ is of class C'. Such a normal solution is *uniquely determined* by prescribing the value of ρ at a single point. This follows from the fact that the condition for $\xi^a - \rho x'^a$ to be normal yields a first order differential equation for ρ. With these facts in mind we can prove the important

LEMMA 8.2. *If* ξ^a *is a solution of equations* (8.1) *having*

$$\xi^a(\tau) = \gamma x'^a(\tau),\ \xi'^a(\tau) = \gamma x''^a(\tau) + \delta x'^a(\tau),$$

at a single point τ, *then* $\xi^a = \rho x'^a$ *on the whole interval* $(t_0 T)$, *where the function* ρ *is of class* D' *and vanishes at the corner parameter values* $t_1, \cdots, t_m$.

Consider an interval containing the point τ, on which the given solution ξ^a is of class C', and consider the associated normal solution $\xi_N{}^a = \xi^a - \rho x'^a$ having $\rho(\tau) = \gamma$ and hence $\xi_N{}^a(\tau) = 0$. The condition $x'^a\xi_N{}'^a = 0$, applied at the point τ, gives $\rho'(\tau) = \delta$, and hence $\xi_N{}'^a(\tau) = 0$. Consequently the normal solution $\xi_N{}^a$ is identically zero and $\xi^a = \rho x'^a$ on the interval in question. Consider next a point τ of discontinuity of ξ'^a, different from $t_1, \cdots, t_m$. At such a point τ the corner conditions (8.3) reduce to

$$(\partial^2 F/\partial x'^a \partial x'^\beta)(\xi'^{\beta-} - \xi'^{\beta+}) = 0.$$

Since $F_1 \neq 0$, the matrix of coefficients has rank $n-1$, and hence $\xi'^{\beta+}$ differs from $\xi'^{\beta-}$ by a multiple of x'^β. Then by the part of the lemma already proved, ξ^a has the form $\rho x'^a$ on the adjacent interval on which ξ^a is of class C'. Finally, suppose that the hypothesis of the lemma holds at $\tau = t_k^-$, corresponding to one side of one of the corners on E_0. By means of the Euler

* Alternative equations equally applicable to our needs would be: $x'^a\xi'^a = \text{constant}$, or, $x'^a\xi^a = \text{constant}$, or, $x'^a\xi^a = \text{linear function of } t$.

† The proof of this is similar to the proof of Lemma 1, page 199, in the paper by Bliss already cited.

equations (2. 1) and the homogeneity condition on the function F, the corner conditions (8. 3) in this case are easily reduced to

$$(8.5) \qquad \gamma(\partial F^-/\partial x^a) - (\partial^2 F^+/\partial x'^a \partial x'^\beta)\xi'^{\beta+} - \gamma(\partial^2 F^+/\partial x'^a \partial x^\beta)x'^{\beta-} = 0.$$

If we multiply these equations by x'^{a+} and sum, we find that the middle term drops out, and

$$\gamma[x'^{a+}(\partial F^-/\partial x^a) - (\partial F^+/\partial x^\beta)x'^{\beta-}] = -\gamma\Omega_0 = 0.$$

Since we have assumed $\Omega_0 \neq 0$ at the corners on E_0, we must have $\gamma = 0$. Substituting this in equations (8. 5), we find that $\xi'^{\beta+}$ must be a multiple of $x'^{\beta+}$, so that if the hypothesis of the lemma holds on one side of a corner, it must hold on the other side also.

9. *A new definition of conjugate point, and the corresponding form of the Jacobi-Caratheodory condition.* A point x is defined to be *conjugate* to a point x_0 on an extremaloid E_0 in case it corresponds to a parameter value $\tau \neq t_0$ such that there exist constants γ, δ, ϵ, and an admissible solution ξ^a of the Jacobi equations (8. 1) with the properties:

1) ξ^a is not of the form $\rho x'^a$;
2) $\epsilon\xi^a(t_0) = 0$, and $\epsilon\xi^a(\tau) = \gamma x'^a(\tau^-) + \delta x'^a(\tau^+)$;
3) γ, δ, ϵ are not all three zero, and $\gamma\delta \geqq 0$.

Here $x'^a(\tau^-)$ and $x'^a(\tau^+)$ refer respectively to the left hand and right hand derivatives of x^a at τ. The special case where γ and δ may both be taken as zero is the only one of interest for a minimizing arc without corners. In case $\epsilon = 0$, it is obvious, since $x'^a x'^a \neq 0$, that $\gamma\delta > 0$ and x is at a corner which is a cusp on E_0.

THE JACOBI-CARATHEODORY CONDITION. *If E_0 is an admissible arc joining the points x_0 and X and minimizing the integral J, and if $F_1 \neq 0$ on E_0 and $\Omega_0 \neq 0$ at the corners on E_0, then there can be no point x conjugate to x_0 on E_0 between x_0 and X.*

Corollary. A minimizing arc can contain no cusp.

The hypotheses imply that E_0 is an extremaloid whose extremal arcs are each of class C''. Moreover, $\Omega_0 < 0$ at the corners, by § 4. Suppose x is conjugate to x_0 and lies between x_0 and X, and let ξ^a be a solution of the Jacobi equations defining the conjugate point x. Let

$$\begin{aligned} \eta^a &= \epsilon\xi^a - \rho x'^a \qquad (t_0 \leqq t \leqq \tau), \\ &= \rho x'^a \qquad (\tau \leqq t \leqq T), \end{aligned}$$

where the function ρ vanishes at $t_0, t_1, \cdots, t_m, T$, except that at $t = \tau$, $\rho(\tau^-) = \gamma$, $\rho(\tau^+) = \delta$. We suppose that ρ is of class D' on $(t_0\,\tau)$ and on $(\tau\,T)$. Then η is an admissible variation, vanishing at t_0 and T, and satisfying the Jacobi equations (8.1) on the intervals $(t_0\,\tau)$ and $(\tau\,T)$. However, η cannot satisfy those equations on the whole interval $(t_0\,T)$, by lemma 8.2, since if $\epsilon \neq 0$, η^a has the form $\rho x'^a$ on only part of the interval, and if $\epsilon = 0$, the parameter value τ corresponds to a corner at which ρ does not vanish. Consequently $J_2(\eta)$ cannot be a minimum. We shall show that $J_2(\eta) \leqq 0$, and hence that E_0 cannot minimize the integral J if E_0 contains a conjugate point x as assumed.

On an interval (t_{k-1}, t_k) not containing τ, the usual manipulations give *

$$\int_{t_{k-1}}^{t_k} 2\omega(t, \eta, \eta')\,dt = \eta^a \frac{\partial \omega(t, \eta, \eta')}{\partial \xi'^a} \Bigg|_{t_{k-1}}^{t_k}.$$

Since η^a vanishes at t_0 and T, and $\partial\omega/\partial\xi'^a$ is continuous except at $t = \tau$, we find

$$\begin{aligned} J_2(\eta) = \int_{t_0}^{T} 2\omega(t, \eta, \eta')\,dt &= \eta^a \frac{\partial \omega(t, \eta, \eta')}{\partial \xi'^a} \Bigg|_{\tau+}^{\tau-} \\ &= \eta^a [\epsilon \frac{\partial \omega(\tau, \xi, \xi')^-}{\partial \xi'^a} - \frac{\partial \omega(\tau, \rho x', \rho x'' + \rho' x')^-}{\partial \xi'^a} \\ &\quad - \frac{\partial \omega(\tau, \rho x', \rho x'' + \rho' x')^+}{\partial \xi'^a}\,]. \end{aligned}$$

Now from equation (8.4) we have

$$(\partial/\partial\xi'^a)\omega(\tau, \rho x', \rho x'' + \rho' x') = \rho(\partial F/\partial x^a).$$

Also $\partial\omega/\partial\xi'^a$ is continuous on the solution ξ. Hence

$$\begin{aligned} J_2(\eta) &= \delta x'^{a+}[\epsilon(\partial/\partial\xi'^a)\omega(\tau, \xi, \xi')^+ - \gamma(\partial F^-/\partial x^a)] \\ &\quad - (\epsilon\xi^a - \gamma x'^{a-})\delta(\partial F^+/\partial x^a) \\ &= \delta\epsilon(\partial F^+/\partial x^\beta)\xi^\beta - \gamma\delta[x'^{a+}(\partial F^-/\partial x^a) - x'^{a-}(\partial F^+/\partial x^a)] \\ &\quad - \delta\epsilon\xi^a(\partial F^+/\partial x^a) \\ &= \begin{cases} 0 \text{ if } x \text{ is not at a corner,} \\ \gamma\delta\Omega_0 \leqq 0 \text{ if } x \text{ is at a corner.} \end{cases} \end{aligned}$$

10. *The relation between the two forms of the Jacobi-Caratheodory condition.* Associated with a set of $2n - 2$ solutions

$$\xi_i^a, \eta_j^a \qquad (i, j = 1, \cdots, n-1),$$

* Cf. Bliss, *loc. cit.*, p. 200.

of the Jacobi equations (8.1) are two important determinants

$$\mathfrak{W}(t) = \begin{vmatrix} \xi_i{}^\alpha & \eta_j{}^\alpha & x'^\alpha & 0 \\ \xi'_i{}^\beta & \eta'_j{}^\beta & x''^\beta & x'^\beta \end{vmatrix},$$

$$\Theta(t, t_0) = \begin{vmatrix} \xi_i{}^\alpha(t) & \eta_j{}^\alpha(t) & x'^\alpha(t) & 0 \\ \xi_i{}^\beta(t_0) & \eta_j{}^\beta(t_0) & 0 & x'^\beta(t_0) \end{vmatrix},$$

and associated with a set of $n-1$ solutions is the determinant

$$D(t) = \begin{vmatrix} \xi_i{}^\alpha & x'^\alpha \end{vmatrix}.$$

We shall proceed to develop the relations between these determinants, the points conjugate to x_0, and the families of extremaloids discussed in § 3.

LEMMA 10.1. *The determination $\mathfrak{W}(t)$ either vanishes identically or is never zero on the interval (t_0, T).*

To prove this, suppose $\mathfrak{W}(\tau) = 0$. Then there exist $2n$ constants a^i, b^i, γ, δ, not all zero, satisfying the equations

$$\begin{aligned} a^i\xi_i{}^\alpha + b^i\eta_i{}^\alpha + \gamma x'^\alpha &= 0 \\ a^i\xi_i{}'^\alpha + b^i\eta_i{}'^\alpha + \gamma x''^\alpha + \delta x'^\alpha &= 0 \end{aligned}$$

for $t = \tau$. Moreover, the constants a^i, b^i, are not all zero, since if they were, we would have $\gamma = \delta = 0$ also. The solution $\xi^\alpha = a^i\xi_i{}^\alpha + b^i\eta_i{}^\alpha$ of the Jacobi equations has $\xi^\alpha(\tau) = -\gamma x'^\alpha(\tau)$, $\xi'^\alpha(\tau) = -\gamma x''^\alpha(\tau) - \delta x'^\alpha(\tau)$. Hence by lemma 8.2, ξ^α has the form $\rho x'^\alpha$ on (t_0, T) and therefore $\mathfrak{W}(t)$ vanishes identically.

LEMMA 10.2. *If the determinant $\mathfrak{W}$ does not vanish, then every solution of the Jacobi equations* (8.1) *is expressible uniquely in the form*

$$\xi^\alpha = a^i\xi_i{}^\alpha + b^i\eta_i{}^\alpha + \rho x'^\alpha,$$

where the a^i, b^i, are constants, and ρ is a function of t of class D' vanishing at $t_1, \cdots, t_m$.

For at an arbitrary point τ of (t_0, T) the equations

$$\begin{aligned} \xi^\alpha &= a^i\xi_i{}^\alpha + b^i\eta_i{}^\alpha + \gamma x'^\alpha \\ \xi'^\alpha &= a^i\xi_i{}'^\alpha + b^i\eta_i{}'^\alpha + \gamma x''^\alpha + \delta x'^\alpha \end{aligned}$$

have unique solutions for the $2n$ unknowns a^i, b^i, γ, δ. Then

$$\eta^\alpha = \xi^\alpha - a^i\xi_i{}^\alpha - b^i\eta_i{}^\alpha$$

is a solution of the Jacobi equations having $\eta^\alpha(\tau) = \gamma x'^\alpha(\tau)$, $\eta'^\alpha(\tau) =$

$\gamma x''^a(\tau) + \delta x'^a(\tau)$, and consequently η^a has the form $\rho x'^a$ on the whole interval (t_0, T), by lemma 8.2.

LEMMA 10.3. *The function $\Theta(t, t_0)$ can be expressed in the form*

$$\Theta(t, t_0) = (t - t_0)^{n-1}\lambda(t, t_0),$$

where λ is continuous at $t = t_0$, and $\lambda(t_0, t_0) = \pm \mathfrak{W}(t_0)$.

Here it is understood that the parameter value t_0 does not correspond to a corner on E_0. Consequently the proof given in Bliss's paper * for his theorem 7 is essentially applicable to this lemma.

LEMMA 10.4. *The function $\Theta(t, t_0)$ vanishes identically if and only if $\mathfrak{W}(t)$ vanishes.*

That the condition $\mathfrak{W}(t) = 0$ is necessary follows at once from the lemmas 10.3 and 10.1. To show $\mathfrak{W}(t) = 0$ is sufficient, we have from the proof of lemma 10.1 that there exist constants a^i, b^i not all zero, and a function $\rho(t)$, such that $a^i\xi_i^a + b^i\eta_i^a + \rho x'^a \equiv 0$. It is an obvious consequence of this that the determinant $\Theta(t, t_0)$ vanishes identically in t and t_0.

LEMMA 10.5. *If the determinant $\Theta(t, t_0)$ does not vanish identically, then the points x conjugate to x_0 on the extremaloid E_0 correspond to the parameter values $t = \tau \neq t_0$ at which Θ vanishes or changes sign.*

By lemmas 10.2 and 10.4, a solution ξ^a of the Jacobi equations determining the conjugate point x is expressible in the form $\xi^a = a^i\xi_i^a + b^i\eta_i^a + \rho x'^a$, where the constants a^i, b^i, are not all zero, and the function ρ vanishes at the parameter values $t_1, \cdots, t_m$, corresponding to the corners. From the conditions $\epsilon\xi^a(\tau) = \gamma x'^a(\tau^-) + \delta x'^a(\tau^+)$, $\epsilon\xi^a(t_0) = 0$, we have the system of linear equations

$$\begin{aligned} &\epsilon a^i\xi_i^a(\tau) + \epsilon b^i\eta_i^a(\tau) + \epsilon\rho(\tau)x'^a(\tau) - \gamma x'^a(\tau^-) - \delta x'^a(\tau^+) = 0 \\ &\epsilon a^i\xi_i^a(t_0) + \epsilon b^i\eta_i^a(t_0) + \epsilon\rho(t_0)x'^a(t_0) = 0. \end{aligned}$$

Consider first the case $\gamma\delta = 0$, in which we have $\epsilon \neq 0$, by § 9. When the point x is not at a corner, the situation may always be reduced to this case. Then obviously Θ must vanish for $t = \tau$, τ^-, or τ^+. In case x is at a corner and $\gamma\delta > 0$, $\rho(\tau) = 0$, we must have

$$\begin{vmatrix} \xi_i^a(\tau) & \eta_i^a(\tau) & [\gamma x'^a(\tau^-) + \delta x'^a(\tau^+)] & 0 \\ \xi_i^a(t_0) & \eta_i^a(t_0) & 0 & x'^a(t_0) \end{vmatrix} = 0,$$

* *Loc. cit.*, p. 206.

from which we see that Θ changes sign for $t = \tau$.

To prove the converse, suppose, in the first place, that Θ vanishes for $t = \tau$. Then the equations

$$\begin{aligned} a^i\xi_i{}^a(\tau) + b^i\eta_i{}^a(\tau) + \gamma x'^a(\tau) &= 0 \\ a^i\xi_i{}^a(t_0) + b^i\eta_i{}^a(t_0) + \delta x'^a(t_0) &= 0 \end{aligned}$$

have solutions a^i, b^i, γ, δ, with the constants a^i, b^i, not all zero. Let $\xi^a = a^i\xi_i{}^a + b^i\eta_i{}^a + \rho x'^a$, where $\rho(t_0) = \delta$, $\rho(\tau) = \rho(t_1) = \cdots = \rho(t_m) = 0$. Then $\xi^a(t_0) = 0$, $\xi^a(\tau) = -\gamma x'^a(\tau)$. By lemmas 10.4 and 10.2, ξ^a is not of the form $\rho x'^a$. Consequently the point x corresponding to $t = \tau$ is conjugate to x_0. Suppose finally that Θ changes sign without vanishing at $t = \tau$. Then $\gamma\Theta(\tau^-, t_0) + \delta\Theta(\tau^+, t_0) = 0$, where $\gamma\delta > 0$. Hence the equations

$$\begin{aligned} a^i\xi_i{}^a(\tau) + b^i\eta_i{}^a(\tau) + \gamma_1[\gamma x'^a(\tau^-) + \delta x'^a(\tau^+)] &= 0 \\ a^i\xi_i{}^a(t_0) + b^i\eta_i{}^a(t_0) + \delta_1 x'^a(t_0) &= 0 \end{aligned}$$

have solutions a^i, b^i, γ_1,δ_1, not all zero. The constant γ_1 is not zero, since otherwise the determinant Θ would vanish at τ. If the constants a^i, b^i, are all zero, we have $\gamma x'^a(\tau^-) + \delta x'^a(\tau^+) = 0$, and the solution ξ^a of the Jacobi equations occurring in the definition of conjugate point may be taken arbitrarily. If the constants a^i, b^i, are not all zero, let $\xi^a = a^i\xi_i{}^a + b^i\eta_i{}^a + \rho x'^a$, where $\rho(t_0) = \delta_1$, $\rho(t_1) = \cdots = \rho(t_m) = 0$. Then ξ^a is a solution of the Jacobi equations, not of the form $\rho x'^a$, having

$$\xi^a(t_0) = 0, \qquad \xi^a(\tau) = -\gamma_1\gamma x'^a(\tau^-) - \gamma_1\delta x'^a(\tau^+), \qquad \gamma_1{}^2\gamma\delta > 0.$$

Suppose we have a one-parameter family of extremaloids $x^a = \phi^a(t, c)$ containing the extremaloid E_0 for $c = c_0$, and with the corners all occurring at the fixed parameter values $t_1, \cdots, t_m$. If the functions ϕ^a are substituted for x^a in the Euler equations (2.1) and the corner conditions (2.2), these equations become identities in t and c. If we differentiate these identities with respect to c and set $c = c_0$, we find that $\xi^a = (\partial/\partial c)\phi^a(t, c_0)$ is a solution of the Jacobi equations (8.1) for the extremaloid E_0. Hence from the $2n - 2$ parameter family of extremaloids $x^a = \Phi^a(t, a, b)$ discussed in § 3 (supposing that the corners all occur at the fixed parameter values t_k), we obtain a set of $2n - 2$ solutions of the Jacobi equations

$$\xi_i{}^a(t) = (\partial/\partial a^i)\Phi^a(t, a_0, b_0), \qquad \eta_i{}^a(t) = (\partial/\partial b^i)\Phi^a(t, a_0, b_0).$$

For this set the determinant $\mathfrak{W}$ does not vanish on at least one extremal arc of E_0, by the imbedding theorems for extremals, and hence $\mathfrak{W}$ does not vanish on E_0, by lemma 10.1. The corresponding determinant Θ does not vanish

identically, by lemma 10.4. As explained in the closing paragraph of § 3, the sign of each of the determinants

$$(10.1)\qquad \left|\begin{array}{cccc} \dfrac{\partial\Phi^a(t)}{\partial a^i} & \dfrac{\partial\Phi^a(t)}{\partial b^i} & \dfrac{\partial\Phi^a(t)}{\partial t} & 0 \\ \dfrac{\partial\Phi^a(t_0)}{\partial a^i} & \dfrac{\partial\Phi^a(t_0)}{\partial b^i} & 0 & \dfrac{\partial\Phi^a(t_0)}{\partial t} \end{array}\right| \begin{array}{l} a=a_0 \\ b=b_0, \end{array}$$

$$(10.2)\qquad \left|\begin{array}{cc} \dfrac{\partial\phi^a}{\partial a^i} & \dfrac{\partial\phi^a}{\partial t} \end{array}\right| \; a=a_0,$$

is unaffected by the transformations necessary to make the corners appear at fixed parameter values. If the parameters a and b are properly selected, the $n-1$ parameter family of extremaloids $x^a=\phi^a(t,a)$ passing through the fixed point x_0 may be obtained from the general family by setting $b^i=b_0{}^i$, i. e., $\phi^a(t,a)=\Phi^a(t,a,b_0)$. Hence the determinant (10.1) is a multiple of the determinant (10.2) with multiplier not zero. Lemma 10.5 combined with these facts yields the

THEOREM. *Let $x^a=\Phi^a(t,a,b)$ be equations of the general $2n-2$ parameter family of extremaloids, and $x^a=\phi^a(t,a)$ be equations of the $n-1$ parameter family passing through the point x_0, in which the extremaloid E_0 is contained for $a=a_0$, $b=b_0$. Then the points x conjugate to x_0 on E_0 correspond to the values of the parameter t for which the determinants* (10.1) *and* (10.2) *vanish or change sign.*

SUFFICIENT CONDITIONS FOR A MINIMUM.

11. *Definition of a field, and a general sufficiency theorem.* A field with m discontinuities* is defined to be a continuum $\mathfrak{F}$ in x-space, having associated with it m corner-manifolds

$$S_k : x^a = X_k{}^a(a)$$

of $n-1$ dimensions, and n slope-functions $p^a(x)$, with the properties:

1) the corner manifolds S_k are non-singular, have no multiple points, do not intersect each other, and each S_k divides $\mathfrak{F}$ into two parts;

2) the slope functions $p^a(x)$ do not vanish simultaneously, and with their first derivatives are continuous between corner manifolds S_k and approach finite limits on each side of each S_k;

* Cf. Bliss, *The Problem of Lagrange in the Calculus of Variations*, mimeographed lectures prepared by O. E. Brown, Chap. IV. Also Bliss, "The Transformation of Clebsch in the Calculus of Variations," *Proceedings of the Toronto Congress*, p. 592.

3) the two limits p^{a-} and p^{a+} of the functions p^a at a point of a manifold S_k always determine directions on the same side of S_k and never tangent to it;

4) the functions $\partial F[x, p(x)]/\partial x'^a$ are continuous in $\mathfrak{F}$;

5) the integral

$$J^* = \int \frac{\partial F[x, p(x)]}{\partial x'^a}\, dx^a$$

is independent of the path in $\mathfrak{F}$.

The condition 3) is equivalent to saying that the two determinants

$$| \partial X_k^a/\partial a^i \quad p^{a-} |, \qquad | \partial X_k^a/\partial a^i \quad p^{a+} |,$$

are both positive or both negative along a corner manifold. It may be proved in the usual way that the solutions of the equations

$$dx^a/dt = p^a(x)$$

between manifolds S_k are all extremals. The condition 4) implies that these extremal arcs may be pieced together to form extremaloids. These are called the *extremaloids of the field.* Along an extremaloid of the field the integrals J and J^* obviously have the same value. From these facts we may readily prove the

GENERAL SUFFICIENCY THEOREM. *Let E be an extremaloid of a field $\mathfrak{F}$ in which the Weierstrassian function*

$$\mathcal{E}[x, p(x), q] \geqq 0$$

for all directions q. Then if C is an admissible curve lying in $\mathfrak{F}$ and joining the same two points as E, we must have

$$J(C) \geqq J(E).$$

If, except for points x on the manifolds S_k, the function $\mathcal{E}$ vanishes only when the direction q coincides with the direction $p(x)$ (i. e., if the extremaloids of the field are positively strong), then $J(C) = J(E)$ only when the curve C coincides with the extremaloid E.

This theorem is applicable even in cases where the hypotheses $F_1 \neq 0$, $\Omega_0 \neq 0$, used in deriving necessary conditions, are not everywhere fulfilled.

12. *Sufficient conditions for the existence of a field.** The theorem we wish to secure is the following:

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* Bolza gives a geometric proof for the plane case in *Vorlesungen über Variationsrechnung*, pp. 381 ff.

Let E_0 be an extremaloid along which $F_1 \neq 0$, and $\Omega_0 \neq 0$ at the corners. If E_0 contains no multiple point and no point conjugate to one of its end-points, then E_0 is an extremaloid of a field $\mathfrak{F}$, as defined in § 11.

From the discussion of § 3 and § 10 we know that E_0 is a member of a $2n-2$ parameter family of extremaloids

$$x^a = \Phi^a(t, a, b),$$

for which the determinant

$$\Theta(t, t_0) = \begin{vmatrix} \dfrac{\partial \Phi^a(t)}{\partial a^i} & \dfrac{\partial \Phi^a(t)}{\partial b^i} & \dfrac{\partial \Phi^a(t)}{\partial t} & 0 \\ \dfrac{\partial \Phi^a(t_0)}{\partial a^i} & \dfrac{\partial \Phi^a(t_0)}{\partial b^i} & 0 & \dfrac{\partial \Phi^a(t_0)}{\partial t} \end{vmatrix}$$

does not vanish nor change sign for $t_0 < t \leqq T$. By lemma 10.3 we have that for t and $\bar{t}_0$ in a sufficiently small neighborhood $t_0 - \gamma \leqq t \leqq t_0 + \gamma$ of t_0,

$$\Theta(t, \bar{t}_0) = (t - \bar{t}_0)^{n-1} \lambda(t, \bar{t}_0), \qquad \lambda(t, \bar{t}_0) \neq 0. \tag{12.1}$$

The function $\Theta(t, \bar{t}_0)$ is continuous for $t_0 - \gamma \leqq \bar{t}_0 \leqq t_0 + \gamma$, and for t in each of the intervals $t_0 + \gamma \leqq t \leqq t_1$, $t_k \leqq t \leqq t_{k+1}$, $(k = 1, \cdots, m;\ t_{m+1} \equiv T)$. Since Θ preserves its sign for this range of t and $\bar{t}_0 = t_0$, Θ will still preserve its sign for a fixed value t_0 such that $t_0 - \gamma < \bar{t}_0 < t_0$, and for the same range of t. The combination of this result with (12.1) shows then that $\Theta(t, \bar{t}_0)$ does not vanish nor change sign for $t_0 \leqq t \leqq T$.

Let $\bar{x}_0$ be the point on E_0 (preceding x_0) corresponding to $\bar{t}_0$, and let

$$x^a = \phi^a(t, a)$$

be equations of the $n-1$ parameter family of extremaloids passing through $\bar{x}_0$ for $t = t_0$, containing E_0 for $a = a_0$. We now suppose also that the parameter t is chosen so that the corners on all the extremaloids occur for the fixed parameter values $t_1, \cdots, t_m$. From the relation between the determinants $\Theta(t, \bar{t}_0)$ and

$$D(t, a) = \left| \partial \phi^a / \partial a^i \qquad \partial \phi^a / \partial t \right|$$

discussed in § 10, we know that $D(t, a_0)$ does not vanish nor change sign for $t_0 \leqq t \leqq T$.

The next step is to show that these facts imply the existence of a unique solution $t = t(x)$, $a^i = a^i(x)$, of the equations $x^a = \phi^a(t, a)$, for points x in a neighborhood of the extremaloid E_0, so that there is a continuum $\mathfrak{F}$ containing the extremaloid E_0 which is in one-to-one continuous correspondence with a neighborhood of the set of points $t_0 \leqq t \leqq T$, $a^i = a_0^i$. The

ordinary implicit function theorems are not immediately applicable, since the derivatives of the functions ϕ^a are not all continuous. Let us recall, however, from the method of constructing the family of extremaloids, that this family may be broken up into $m+1$ families of extremals

$$x^a = \psi_k{}^a(t, a) = \phi^a(t, a) \qquad (t_{k-1} \leqq t \leqq t_k).$$

Moreover, these extremals may be slightly extended beyond the corner manifolds so that the functions $\psi_k{}^a$ have first and second partial derivatives which are continuous, and the determinants

$$D_k(t, a) = \left| \partial\psi_k{}^a/\partial a^i \quad \partial\psi_k{}^a/\partial t \right|$$

do not vanish and all have the same sign. The standard implicit function theorems are therefore applicable to each of these families. For points x near a corner x_k, there will be a unique solution $t^-(x)$, $a^{i-}(x)$ near t_k, $a_0{}^i$, of the equations $x^a = \psi_k{}^a(t, a)$, and a unique solution $t^+(x)$, $a^{i+}(x)$, of the equations $x^a = \psi^a{}_{k+1}(t, a)$. If either t^- or t^+ equals t_k, then $t^- = t^+ = t_k$, $a^{i-} = a^{i+}$, since

$$\psi_k{}^a(t_k, a) = \psi^a{}_{k+1}(t_k, a). \tag{12.2}$$

The remaining cases are:

1) $t^- < t_k, \quad t^+ < t_k;$
2) $t^- > t_k, \quad t^+ > t_k;$
3) $t^- < t_k, \quad t^+ > t_k;$
4) $t^- > t_k, \quad t^+ < t_k.$

We shall show that in a sufficiently small neighborhood of (x_k, t_k, a_0) the third and fourth cases cannot occur, so that the equations $x^a = \phi^a(t, a)$ define functions $t(x)$, $a^i(x)$, which are single-valued near x_k. In the determinants D_k, D_{k+1}, suppose the arguments (t, a) of each element to be replaced by independent variables, and denote the resulting functions of n^2 variables by $\mathfrak{D}_k$, $\mathfrak{D}_{k+1}$. Since $\mathfrak{D}_k = D_k(t_k, a_0)$, $\mathfrak{D}_{k+1} = D_{k+1}(t_k, a_0)$, when the arguments of all the elements are set equal to (t_k, a_0), there will be a neighborhood of (t_k, a_0) in which $\mathfrak{D}_k$ and $\mathfrak{D}_{k+1}$ do not vanish and have the same sign. Suppose that either the third or the fourth case listed above occurs in this neighborhood. Then by using equations (12.2) we have

$$\begin{aligned} 0 &= \psi^a{}_{k+1}(t^+, a^+) - \psi_k{}^a(t^-, a^-) \\ &= \psi^a{}_{k+1}(t^+, a^+) - \psi^a{}_{k+1}(t_k, a^+) \end{aligned} \tag{12.3}$$

$$+\psi_k{}^a(t_k, a^-) - \psi_k{}^a(t^-, a^-)$$

$$+\psi_k{}^a(t_k, a^+) - \psi_k{}^a(t_k, a^-)$$

$$= \left[\frac{\partial \psi^a{}_{k+1}(\bar{t}^+, a^+)}{\partial t}\left(\frac{t^+ - t_k}{t^+ - t^-}\right) + \frac{\partial \psi_k{}^a(\bar{t}^-, a^-)}{\partial t}\left(\frac{t_k - t^-}{t^+ - t^-}\right)\right](t^+ - t^-)$$

$$+ \frac{\partial \psi_k{}^a(t_k, \bar{a})}{\partial a^i}(a^{i+} - a^{i-}),$$

where the arguments $\bar{t}^+$, $\bar{t}^-$, $\bar{a}$, are those prescribed by the theorem of the mean. From (12.2) we have also

$$\frac{\partial \psi_k{}^a(t_k, a)}{\partial a^i} = \frac{\partial \psi^a{}_{k+1}(t_k, a)}{\partial a^i},$$

so that the determinant of the matrix of coefficients of the quantities $(t^+ - t^-)$, $(a^{i+} - a^{i-})$ in equations (12.3) reduces to

$$\left(\frac{t^+ - t_k}{t^+ - t^-}\right)\mathfrak{D}_{k+1} + \left(\frac{t_k - t^-}{t^+ - t^-}\right)\mathfrak{D}_k \neq 0.$$

Hence the third and fourth cases cannot occur. The extended implicit function theorem of Bolza can now be proved for the present situation by exactly the same arguments as those used by Bliss for the case of continuous derivatives.*

We can now define the slope functions for the field by the equations

$$p^a(x) = (\partial/\partial t)\phi^a[t(x), a(x)].$$

The five properties in the definition of a field in § 11 are readily verified. The third follows from the fact that the determinant $D(t, a)$ does not vanish nor change sign in a sufficiently restricted neighborhood of the extremaloid E_0. To verify property 5), the invariance of the Hilbert integral, we compute the partial derivatives of the *field-integral*

$$W(x) = \int_{t_0}^{t(x)} F\{\phi[t, a(x)],\ (\partial/\partial t)\phi[t, a(x)]\}dt$$

and find that between corner manifolds they are

$$\partial W/\partial x^a = (\partial/\partial x'^a)F[x, p(x)].$$

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* Cf. *Princeton Colloquium Lectures*, pp. 20, 21.

Since these derivatives are continuous even on the corner manifolds, we see at once that the Hilbert integral J^* is independent of the path, for all admissible curves in $\mathfrak{F}$.

13. *A set of sufficient conditions for a relative minimum.* From the preceding sections we can now readily prove that the following conditions are sufficient for $J(E_0)$ to be a *proper strong relative* minimum.*

I. *E_0 is an extremaloid having no multiple point.*

II. *$\Omega_0 \neq 0$ at the corners on E_0.*

III. *E_0 is positively strong, and is uniquely determined by each of its elements (x, x').*†

IV. *E_0 contains no point conjugate to one of its end-points.*

For, the third condition implies $F_1 \neq 0$ along E_0, and this with the other conditions implies that E_0 is an extremaloid of a field, by § 12. Then we apply lemmas 5. 3 and 5. 6 to show that the extremaloids of the field are all positively strong if the field is restricted to a sufficiently small neighborhood of E_0. This enables us to apply the general sufficiency theorem of § 11.

The third condition is understood to mean that the function $\mathcal{E}_1$ of § 5 does not vanish at either of the end-points x_0, X, of the extremaloid E_0.

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* For the terminology, cf. Bolza, *loc. cit.*, pp. 17, 197.

† See § 5, p. 11.

Geodesics on a Toroid.*

By B. F. Kimball.

1. *Introduction.* The study of the geodesics on a toroid is of interest, not only because of the geometric properties of the surface which this study discloses, but also because the geodesic curves are the solutions of a differential equation which is of interest; namely,

$$v'(u) = [r(v)/\alpha]\,[r^2(v) - \alpha^2]^{\frac{1}{2}}$$

where α is a constant not zero and $r(v)$ is a real, analytic, periodic function of v greater than zero. The function $r(v)$ is restricted to have only one maximum and one minimum value in a period interval. Taking $v = 0$ at a maximum of r it is also required that $r(v) = r(-v)$.

The results obtained in this paper through Part III apply to solutions of the above differential equation, the solutions being mapped on a surface of revolution. The results set forth in Part IV apply to a slightly more restricted differential equation [see condition (c) § 2]. The reason for introducing such further restriction is that a more complete and interesting study of the envelopes of the geodesics on the surface is thereby made possible.

The surface of revolution defined in § 2 is called a toroid since it resembles a torus. The torus was defined by Cayley † as "a surface generated by the rotation of a conic about a fixed axis anywise situate." The geodesics on the anchor ring torus have already been studied by G. A. Bliss.‡

The differential equation of the geodesics on the anchor ring, Bliss solved by means of elliptic functions and in an elegant manner studied the solutions thus obtained. A destinctive feature of the present paper is the policy of dealing directly with the differential equations, since closed solutions are not obtainable in this more general case, and of getting at the desired properties of the solutions directly from them. This paper uses for the first time, as far as the writer is aware, the equation of second variation as an aid in a specific investigation of the envelopes of geodesics, and makes application of the envelope theory of Lindeberg (cf. § 16) to that study. This introduction

* Presented to the American Mathematical Society, October 30, 1926.

† Cayley, *Mathematical Papers*, Vol. 7, p. 246, and Vol. 8.

‡ "The Geodesic Lines on the Anchor Ring," *Annals of Mathematics*, Ser. 2, Vol. 4 (1903), p. 1.

would not be complete without making an acknowledgment of the inspiration gained from the work of Prof. Marston Morse in this field of study.

PART I. THE SURFACE.

2. *Definition of the surface.*

(a) *The surface is taken as a surface of revolution with the z-axis as its axis of revolution. The generatrix is a simple, closed curve which lies initially in the xz-plane and does not cut the z-axis. It will be represented in terms of two functions $r(v)$ and $z(v)$ which are real, analytic, periodic functions of v for all real values of v and with a period 2ω. In its initial position it is expressed in the form*

$$(2.1) \qquad \begin{aligned} x &= r(v) & r(-\omega) &= r(+\omega), \\ z &= z(v) & z(-\omega) &= z(+\omega), \end{aligned}$$

where v is the arc length of the curve. Only values of v in the interval $-\omega \leqq v \leqq +\omega$ will be considered.

(b) *It shall further be required that the curve (2.1) be symmetrical with respect to the x-axis.* Of the intersections of the generating curve with the x-axis, let A be the one of maximum x and B the one of minimum x. The points A and B divide the generatrix into two segments. Of these two segments at least one has some points at which z is positive. Let C be a point on this segment at which z takes on its absolute maximum. Let D be the reflection of C in the x-axis. The arc length v used in (2.1) will be measured in the sense $A\,C\,B$ from A as origin.

(c) The last important restriction on the above surface is that *the total curvature K of the surface satisfy the relation $dK/dv < 0$ at all points of $A\,C\,B$ except at A* and *B.*

3. *Some properties of the surface.* One can write the equations of the surface in the form:

$$x = r(v)\cos u, \quad y = r(v)\sin u, \quad z = z(v),$$

where u is the angle that the half meridian plane through the point makes with the xz-plane measured in the xy-plane from the x-axis in a sense which is counterclockwise from the point of view of a position on the positive *z-axis.* Then the square of the element of arc on the surface will be

$$(3.1) \qquad ds^2 = dv^2 + r^2(v)\,du^2.$$

It follows from a familiar formula of Differential Geometry * that

* Eisenhart, *Differential Geometry* (1909), p. 208, VI (55).

$$K = -r''(v)/r(v). \tag{3.2}$$

Let k be the curvature of the generating curve taken with the sign given by the curvature formula:

$$k = r'z'' - z'r''. \tag{3.3}$$

Also since v is the arc length

$$r'^2 + z'^2 = 1. \tag{3.4}$$

This gives

$$r'r'' + z'z'' = 0. \tag{3.5}$$

Multiplying (3.3) by $-z'$, and substituting for $z'z''$ its value obtained from (3.5) one obtains

$$-z'k = r'' \tag{3.6}$$

and (3.2) becomes

$$K = z'k/r. \tag{3.7}$$

From hypothesis (a), (b) and (c) and the above formulae one obtains the following theorem concerning the properties of the generating curve. The proof will follow.

THEOREM 1. (a) *The curvature k of the generatrix is always positive.* (b) *The derivative $r'(v)$ vanishes only twice on the generatrix; namely, at the points A and B and at these points r takes on its absolute maximum and its absolute minimum respectively.*

(a) The proof of statement (a) follows from the consideration of the formula (3.7). Since $dK/dv < 0$ between A and B one sees that K cannot vanish more than once on $A\,C\,B$ (including the end points A and B). Hence from the above formula z' cannot vanish more than once. But z' necessarily vanishes at the maximum C. Thus K is zero on $A\,C\,B$ only at the point C. It then follows that if k vanishes at all on $A\,C\,B$, it can only vanish at C. One can now show that k cannot vanish at this point. Differentiating (3.7) and employing hypothesis (c) one obtains the relation

$$\frac{dK}{dv} = \frac{r(z'k' + z''k) - z'kr'}{r^2} < 0, \tag{3.8}$$

which shows that k and z' cannot vanish at the same point. Thus k does not vanish at C and from symmetry cannot vanish at any point on the segment $A\,D\,B$. Thus statement (a) is proved. To determine the sign of k observe that, at the point A,

$$z' = 1, \quad r' = 0, \quad r'' < 0.$$

Formula (3.3) then shows that k is positive at A and hence positive at all points of the curve.

(b) The statement (b) can be proved as follows. Consider the generatrix in the xz-plane. Let θ denote the angle which the tangent directed in the sense of increasing v makes with the positive x-axis. For a simple closed curve possessing a continuously turning tangent at every point it has been proved * that this angle changes by a net amount of 2π as the curve is traversed once. Now for above curve $k = d\theta/dv > 0$. Hence θ must increase monotonically, and by an amount 2π as the curve is traversed from D to D in the positive sense. Thus $\cos\theta$ is zero only twice; namely, at A and B. The statement (b) above follows.

If the generatrix is an ellipse with axis AB equal to $2a$ and axis CD equal to $2b$ and r_0, the distance from the center of the ellipse to the z-axis, greater than a; it can be shown that necessary and sufficient conditions that $dK/dv < 0$ for the interval $0 < v < \omega$ are that

$$a \geqq b,$$

or

$$(3.9) \qquad a < b < (2/3^{\frac{1}{2}})a, \quad r_0 \geqq 4a(b^2 - a^2)/(4a^2 - 3b^2).$$

PART II. THE GEODESICS.

4. *The differential equations of the geodesics.* It was seen that for the system of coordinates chosen, the square of the element of arc on the surface was

$$(4.1) \qquad ds^2 = dv^2 + r^2(v)\,du^2.$$

The differential equation of the geodesics in parametric form is thus †

$$(4.2) \qquad (\dot{u}\,\ddot{v} = \ddot{u}\,\dot{v}) - (2\,r'/r)\dot{u}\,\dot{v}^2 - r\,r'\,\dot{u}^3 = 0$$

where a dot above a letter indicates differentiation with respect to the parameter t and r' is dr/dv. One also needs the equation of the geodesics in non-parametric form for the case where u is taken as the independent variable. Setting $t = u$, equation (4.2) becomes

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* Watson, *Proceedings of the London Mathematical Society*, Ser. 2, Vol. 15 (1916), pp. 227-42.

† Bolza, *Variationsrechnung* (1909), § 26, p. 210. Future references to this book will be indicated by the letter B.

$$v''(u) - [2r'(v)/r(v)]\, v'^2(u) - r(v)r'(v) = 0. \tag{4.3}$$

Other formulae for the geodesics are found by integrating one of the Euler equations giving

$$r^2\dot{u}/[\dot{v}^2 + r^2(v)\dot{u}^2]^{\frac{1}{2}} = \alpha, \tag{4.4}$$

where α is a constant of integration. Write (4.4) in the form

$$r^2 du/ds = \alpha \tag{4.5}$$

or

$$r\cos\chi = \alpha \tag{4.6}$$

where χ is that angle which the positive direction of a geodesic makes with the positive tangent to the parallel thru the point on the geodesic considered, measured in a counterclockwise sense from the point of view of a position on the exterior normal to the surface. For geodesics along which u may be taken as the independent variable, when one sets $t = u$ (4.4) becomes

$$r^2/(v'^2 + r^2)^{\frac{1}{2}} = \alpha, \tag{4.7}$$

which solved for v' gives

$$v' = \pm\,(r/\alpha)\,(r^2 - \alpha^2)^{\frac{1}{2}}. \tag{4.8}$$

Now the equation of a geodesic must be a solution of (4.4) and therefore of equations (4.5) — (4.8) (cf. B. § 26); but all curves that are solutions of these equations are not geodesics—*e. g.*, any parallel will be a solution of (4.6), while in order that it be a solution of the general equation (4.2) it is necessary that r' be zero. Incidentally it is seen that *the only parallels that are geodesics are the inside and outside equators, that is, the curves* $v = 0$ *and* $v = \omega$.

5. *The symmetry of the geodesics.* Taking note of the fact that $r(v)$ is an even function of v, it is seen from the form of the equation (4.2) that

(5.1) *the system of geodesics is symmetrical with respect to the equatorial plane of the surface;*

(5.2) *it is symmetrical with respect to any meridian plane;*

(5.3) *if the surface be rotated about an axis of revolution through any angle, each geodesic on the surface in its original position will coincide at the end of the rotation with a geodesic on the surface in its final position.*

6. *The classification of the geodesics.* Consider the geodesics issuing from a given point (u_0, v_0) at an angle χ restricted to the values

$$-\pi/2 < \chi \leqq +\pi/2. \tag{6.1}$$

Clearly a reflection of these geodesics in the meridian plane through (u_0, v_0) will give all the other geodesics through (u_0, v_0). Let the maximum and minimum values of r on the generatrix be respectively r_2 and r_1. The geodesics issuing from (u_0, v_0) with the angles χ given by (6.1) will determine values of α in (4.6) such that $0 \leqq \alpha \leqq r_2$. These geodesics will be classified as follows:

Class I	$\alpha = 0,$
Class II	$0 < \alpha < r_1,$
Class III	$\alpha = r_1,$
Class IV	$r_1 < \alpha < r_2,$
Class V	$\alpha = r_2.$

7. *The nature of the geodesics.* The following theorems may be verified by using (4.6), (4.8) and the well-known fact that there is a unique geodesic through a given point in a given direction on the surface. In the case of Theorem 5 formula (4.3) will also be needed. Since the results are not essentially new the proofs of these theorems are omitted.*

THEOREM 2. *The meridians ($u = const.$) are the only geodesics which belong to Class I.*

THEOREM 3. *On a geodesic of Class II, v must increase or decrease without limit as u increases without limit, and the geodesic will cross both inside and outside equators an infinite number of times.*

THEOREM 4. *Geodesics of Class III consist of (A) the inside equator, and (B) two geodesics through every point of the surface which approach the inside equator asymptotically in either sense but never cross it. The cosine of the angle χ_1 at which a geodesic of Class III(B) crosses the outside equator is given by the formula* $\cos \chi_1 = r_1/r_2$.

THEOREM 5.† *Each geodesic of Class IV lies in the region on the surface for which $r \geqq \alpha$ and becomes successively tangent to the two parallels $r = \alpha$ as u increases.*

THEOREM 6. *The only geodesic in Class V is the outside equator.*

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* Forsythe, *Differential Geometry* (1912), § 93, p. 132.

† For a further description of geodesics of Class IV see Theorem 14.

PART III. THE CONJUGATE POINTS.

8. *The formulae needed for the discussion of the conjugate points.* There will be occasion to speak of the rth conjugate point to a point P on a given geodesic E. Let the coordinates of this point be (u_0, v_0). Consider a solution $\bar{w}(u)$ of the Jacobi differential equation (B. § 29) set up for the geodesic E such that

$$\bar{w}(u_0) = 0, \qquad \bar{w}'(u_0) = 1.$$

From the theory of the second order differential equation * it is known that the zeros of $\bar{w}(u)$ are discrete. The first conjugate point to P on E is the zero of $\bar{w}(u)$ corresponding to the smallest value of u greater than u_0, the second conjugate point to P on E is the second zero of $\bar{w}(u)$ for u greater than u_0, and in general the rth conjugate point to P is the rth zero of $\bar{w}(u)$ on E for u greater than u_0. The words "conjugate point" will be used in this article to designate the first conjugate point as defined above. Conjugate points denoted by $P_1, P_2, \cdots, P_r$, will refer to the first, second, $\cdots$, rth conjugate points according as the subscripts are $1, 2, \cdots, r$.

It will be convenient to make use of the following facts derivable from the general theory of the Calculus of Variations. If there be given any geodesic segment G without double points on the surface, it is possible (B. § 40) to choose a set of surface coordinates (s, n) as follows. The curves $s = \text{const.}$ may be taken as geodesics perpendicular to G, and n can be taken as the distance measured along these geodesics from G in one of the two possible senses chosen arbitrarily, while s is the distance measured along G from a given point in a given sense. The domain in which it is possible to take such a set of coordinates and still retain the one-to-one correspondence between the pairs (s, n) and the points on the surface will in general have to be limited to a sufficiently small region R neighboring G, but including G entirely within the interior. For the coordinates (s, n) the first fundamental quadratic form of the surface will be

$$d\sigma^2 = dn^2 + C^2(s, n)\,ds^2 \tag{8.1}$$

(B. § 28) where $C(s, n)$ is a real positive analytic function of s and n in R and

$$C(s, 0) = 1, \quad C_n(s, 0) = 0, \quad C_s(s, 0) = 0. \tag{8.2}$$

Under such a choice of a coordinate system consider an analytic family of geodesics

$$n = \Phi(s, \alpha), \quad 0 = \Phi(s, 0), \tag{8.3}$$

* Bôcher, *Leçons sur les Méthodes de Sturm*, § 12, p. 43.

where Φ is analytic in s and α in the neighborhood of a segment G of the geodesic $\alpha = 0$. Let $K(s, n)$ denote the total curvature of the surface as a function of s and n. The Jacobi differential equation, otherwise known as the equation of first variation,* for this geodesic G will then take the form

$$w''(s) + K(s, 0)w(s) = 0. \tag{8.4}$$

It can be shown that the equation of the second variation (B. § 8b) will be

$$\Phi''_{aa}(s, 0) + K(s, 0)\Phi_{aa}(s, 0) = -K_n(s, 0)\Phi_a^2(s, 0), \tag{8.5}$$

and both of these equations will hold at all points of the segment G.

9. *Conjugate points on geodesics of Class I.* The only geodesics of this class are the meridian ovals. *No point on a meridian has a conjugate point.*

Along any meridian one can regard v as the independent variable. The complete set of meridian ovals is a family of geodesics given by the equation

$$u = \Phi(v, \beta)$$

where $\Phi(v, \beta)$ has the simple form

$$\Phi(v, \beta) = \beta,$$

and β is an arbitrary constant. The Jacobi equation in the nonparametric form with v the independent variable and w the dependent variable can be set up for any particular meridian and according to the general theory it will have for one particular solution

$$w = \Phi_\beta(v, \beta) = 1.$$

Thus this particular solution never vanishes. Therefore by Sturm's separation theorem (B. § 11) the solution of the Jacobi equation which determines the conjugate point P_1 to a given point P on the meridian considered cannot vanish twice. Hence no such conjugate point to P exists on a meridian. It may be noted that the above proof would apply to any analytic, regular surface of revolution for which the generatrix is a closed curve.

10. *Conjugate points on geodesics of Class II. There are no conjugate points on geodesics of this class.*

Any geodesic of this class can be represented in the form $v = v(u)$, where $v(u)$ is an analytic function of u for all real values of u. Symmetry considerations show that the geodesics obtained by rotating the given geo-

* Darboux, *Théorie des Surfaces*, Vol. 3, § 627, p. 97.

desic through any angle about the axis of revolution of the surface will also be geodesics of Class II. The family of geodesics so obtained can be represented as follows:

$$v = \Phi(u, \beta) = v(u + \beta),$$

where β is an arbitrary constant. The Jacobi equation of the nonparametric case set up for any one of these geodesics will have as a particular solution

$$\Phi_\beta = v'(u + \beta).$$

According to formula (4. 8) this derivative is never zero for a geodesic of Class II. Again applying Sturm's separation theorem one obtains the result to be proved.

11. *Conjugate points on geodesics of Class III.*

(A) *There are no conjugate points corresponding to points on the inside equator.* Reference to formula (3. 7) shows that the surface curvature is a negative constant on the inside equator. Accordingly a solution of the Jacobi equation (8. 4) which vanishes once and is not identically zero never vanishes again. Thus there are no conjugate points corresponding to points on the inside equator.

(B) Geodesics of Class III (B) are geodesics asymptotic to the inside equator. *These geodesics have no conjugate points.* This can be proved by a method similar to that used in the proof of the non-existence of conjugate points on geodesics of Class II. Thus the theorem:

THEOREM 7. *No point on a geodesic of Class I, II, or III has a conjugate point.*

12. *Conjugate points on geodesics of Class IV. On a geodesic of Class IV the rth extremum of v following a given extremum of v is the rth conjugate point to that extremum.* To prove this statement one proceeds as in the discussion of geodesics of Classes II and III(B). If the given geodesic is represented in the form $v = v(u)$, then

$$v = v(u + \beta)$$

will represent a family of geodesics of this class, while $v'(u)$ will be a solution of the Jacobi equation corresponding to the given geodesic. This solution $v'(u)$ vanishes at the extrema of $v(u)$ and only at these extrema, and the statement is proved.

The following lemma of the Calculus of Variations will be needed:

LEMMA 1. *If Q_1, Q_r and Q_{r+1} are respectively the first, rth and $r+1$th conjugate points (cf. § 8) to the point Q on a given geodesic E corresponding to a regular problem in the Calculus of Variations, then as a point P varies on E from Q to Q_1 moving always in the same sense, its rth conjugate point P_r will move continuously from Q_r to Q_{r+1}, moving always in the same sense.*

In order to prove this lemma, take s as the arc length measured along the given geodesic in the sense that leads from Q to Q_1. Let $w_1(s)$ and $w_2(s)$ be any two independent solutions of the Jacobi equation. Let s and s_r denote the values of s at P and P_r and denote by a and a_r the values of s at Q and Q_r respectively. Then

$$\begin{vmatrix} w_1(s) & w_2(s) \\ w_1(s_r) & w_2(s_r) \end{vmatrix} = 0, \tag{12.1}$$

and if one shows that equation (12.1) can be solved for s_r as a continuous, monotonically increasing function of s, for s in the interval

$$a \leqq s \leqq a_1 \tag{12.2}$$

corresponding to the extremal from Q to Q_1, the lemma will be demonstrated. Regard the determinant (12.1) as a function $\phi(s, s_r)$. Consider a particular pair of the conjugate points $(s^0, s_r{}^0)$ which is thus a solution of $\phi(s, s_r) = 0$, where s^0 is a value of s in the interval (12.2). In order to show that

$$\phi_{s_r}(s^0, s_r{}^0) = \begin{vmatrix} w_1(s^0) & w_2(s^0) \\ w'_1(s_r{}^0) & w'(s_r{}^0) \end{vmatrix} \neq 0, \tag{12.2}$$

observe that

$$\begin{vmatrix} w_1(s^0) & w_2(s^0) \\ w_1(s) & w_2(s) \end{vmatrix}$$

may be regarded as a function $u(s)$ which with s^0 constant is a solution of the Jacobi equation that vanishes for $s = s_r{}^0$. Moreover $u(s)$ is not identically zero, for if it were, $w_1(s)$ and $w_2(s)$ would be linearly dependent. Furthermore $u'(s_r{}^0)$ cannot then be zero since $u(s_r{}^0) = 0$. But $u'(s_r{}^0)$ is the determinant in (12.3). It is thus proved not zero. Accordingly (12.1) can be solved for s_r as a continuous function of s in the interval $a \leqq s \leqq a_1$. By Sturm's separation theorem it follows that P_r moves always in the same sense from Q_r to Q_{r+1} as P moves from Q to Q_1 in one sense. The lemma is thus proved. From Lemma 1 and the statement proved at the beginning of this section we have the following theorem:

THEOREM 8. *On a geodesic of Class IV the rth extremum of v following a given extremum of v is the rth conjugate point to that extremum. As a point P moves along the geodesic from one extremum to the next following P, its rth conjugate point P_r moves continuously and in the same sense from the rth extremum following P to the $r+1$th.*

13. *Conjugate points on geodesics of Class V.*

THEOREM 9. *The angular measure, d, from a point on the outside equator to its first conjugate point is given by the formula*

$$d = \pi/(r_2 k)^{\frac{1}{2}}, \tag{13.1}$$

where k is the curvature of the generatrix [*cf.* (3.3)].

This theorem can be easily proved by setting up the Jacobi equation (8.4) for the outside equator and noting that $\bar{K}(s) = k/r_2$.

If the generatrix of the surface were an ellipse with axes $2a$ and $2b$ respectively parallel to the x and z coordinate axes one finds that

$$d = \pi b/(r_2 a)^{\frac{1}{2}},$$

and by taking a large enough relative to b, d can be made as small as desired. Under the initial hypothesis [cf. (3.9)] $b < (2/3^{\frac{1}{2}})a$. Since $r_2 > 2a$ it follows that $d < b\pi/2^{\frac{1}{2}}a < (2/3)^{\frac{1}{2}}\pi$. *Thus d is never as great as* $(2/3)^{\frac{1}{2}}\pi$ *for the surface when the generatrix is an ellipse.*

PART IV. THE ENVELOPES OF THE GEODESICS.

14. *The integral of the equation of second variation.* Let there be given a geodesic of Class IV or V. Let the points neighboring this geodesic be referred to normal geodesic coordinates (s, n) (cf. § 8) so that the given geodesic is the curve $n = 0$. Let s_0 be the s coordinate of a point P on this geodesic. Furthermore, let there be given an analytic family of geodesics neighboring $n = 0$ through P for which

$$n = \Phi(s, a), \qquad a \equiv \Phi_s(s_0, a). \tag{14.1}$$

Then

$$\Phi(s_0, a) \equiv 0, \qquad \Phi(s, 0) \equiv 0, \tag{14.2}$$

and the given geodesic $n = 0$ is given by $a = 0$. Let $\bar{w}(s)$ denote the solution of the Jacobi equation [cf. (8.4)] which satisfies the conditions

$$\bar{w}(s_0) = 0, \qquad \bar{w}'(s_0) = 1.$$

Denote by $z(s)$ the function $\Phi_{aa}(s, 0)$. Then $z(s)$ satisfies the differential equation

$$z'' + K(s,0)z = -\bar{w}^2(\partial/\partial n)K(s,0) \tag{14.3}$$

[cf. (8.5)], where the initial conditions $z(s_0) = z'(s_0) = 0$ obtain. These initial conditions follow from the identities in (14.1) and (14.2). In order to discuss the envelope of geodesics of Classes IV-V a formula for $z(s)$ at the rth conjugate point $(s_r, 0)$ to P on the geodesic $n = 0$ will be needed.

LEMMA 1. *At s_r, the rth conjugate point to s_0 on $a = 0$,*

$$z(s_r) = \Phi_{ca}(s_r, 0) = (-1)^r A \int_{s_0}^{s_r} \bar{w}^3(t) \frac{\partial K}{\partial n}(t, 0)\, dt \tag{14.4}$$

where A is a positive constant.

In order to prove this lemma one notes that

$$\bar{w}'' + K(s,0)\ \bar{w} = 0. \tag{14.5}$$

Multiply (14.3) by $-\bar{w}$ and (14.5) by $+z$ and add the two equations. The result is the equation

$$z\bar{w}'' - \bar{w}z'' = \bar{w}^3\ (\partial/\partial n)K(s,0).$$

Integrating from s_0 to s_r one obtains

$$z\bar{w}' - \bar{w}z' \Big]_{s_0}^{s_r} = \int_{s_0}^{s_r} \bar{w}^3(t) \frac{\partial K}{\partial n}(t,0)\, dt.$$

At s_0 and its rth conjugate point s_r, $\bar{w}$ is zero. Moreover $z(s_0)$ is zero. Accordingly

$$z(s_r)\bar{w}'(s_r) = \int_{s_0}^{s_r} \bar{w}^3(t) \frac{\partial K}{\partial n}(t,0)\, dt.$$

Now $\bar{w}'(s_r)$ is positive or negative according as r is even or odd. Hence formula (14.4) follows.

15. *The existence theorem for the rth envelope.* In order to establish an existence theorem for the rth envelope a lemma will be needed, concerning the zeros of a solution of the second order differential equation

$$u''(x) + p(x,\lambda)u'(x) + q(x,\lambda)u(x) = 0 \tag{15.1}$$

where p and q are analytic functions of x and λ in the domain

$$-d_1 \leqq \lambda \leqq d_1 \qquad d_1 > 0 \tag{15.2}$$

$$x_0 \leqq x \leqq x_0 + d_2 \qquad d_2 > 0. \tag{15.3}$$

It is known from the general theory of the second order differential equation that for any value of the parameter λ the zeros of a solution of this equation which is not identically zero are discrete. Consider those solutions $u(x, \lambda)$ such that for $x = x_0$ and any value of λ in (15.2)

$$\bar{u}(x_0, \lambda) = 0, \qquad \bar{u}_x(x_0, \lambda) = 1. \tag{15.4}$$

It will be assumed that for every value of λ in the interval (15.2) there are r values of x that make $\bar{u}(x, \lambda)$ zero in the interval (15.3). Order these r values of x for a given λ according to increasing magnitude and denote by $X(\lambda)$ the rth value of x thus obtained.

LEMMA 1. *The function $X(\lambda)$ which gives the rth zero of the solution $\bar{u}(x, \lambda)$ is a single valued analytic function of λ in the interval* (15.2).

This lemma can be proved from the well-known implicit function theorems. The proof is not given here. Take a point P with coordinates (u_0, v_0). Let there be given a system of geodesics on the surface expressible in the form

$$v = f(u, \beta) \tag{15.5}$$

$$\beta = f_u(u_0, \beta) \tag{15.6}$$

through P, where f is an analytic function of u and β for all values of u, and for β satisfying

$$-d \leqq \beta \leqq +d \qquad d > 0. \tag{15.7}$$

Let the rth conjugate points to P exist on all these geodesics for β in (15.7). Take

$$u_r = U_r(\beta), \qquad v_r = V_r(\beta) \tag{15.8}$$

as the u and v coordinates of the rth conjugate point to (u_0, v_0) on the geodesic β of (15.5).

THEOREM 10. *The functions $U_r(\beta)$ and $V_r(\beta)$ will be analytic for β in the interval* (15.7) *and will give an envelope to the system of geodesics* (15.5) [*or a point through which all the geodesics* (15.5) *pass*]. *Each of these geodesics will be tangent to the envelope at the rth conjugate point to P on that geodesic.*

Let the Jacobi equation be set up for each geodesic of the system (15.5), corresponding to the Calculus of Variations problem of minimizing the integral

$$[r^2(v) + (dv/du)^2]^{\frac{1}{2}}.$$

This equation is a second order differential equation expressible with coeffi-

cients which are analytic functions of independent variable u and parameter β. Identify u and β of this Jacobi equation respectively with x and λ of (15.1). Lemma 1 then shows that $U_r(\beta)$ is an analytic function of β in the interval (15.7). Now substitute $U_r(\beta)$ for u in (15.5) and one finds that $V_r(\beta)$ is also analytic in the interval (15.7). That the curve thus determined by equations (15.8) satisfies the usual properties of the envelope as stated in the latter part of the theorem, or else gives a point through which pass all the geodesics of (15.5), follows as proved in Bolza for the "Im Kleinem" theorem (B. § 47(a), p. 358).

16. *The form of the envelope in the neighborhood of a conjugate point.* Let C be a given geodesic. For the sake of clearness in the geometric interpretation of the results take orthogonal geodesic coordinates (cf. § 8) such that $n = 0$ gives the geodesic C. The geodesic C will be called the central geodesic. The relative orientation of positive sense of measurement of s and n will be taken so that from the point of view of the external normal to the surface it will be the same as that of u and v, respectively, for a point on the outside equator. Take a point P on C with coordinates $(s_0, 0)$ and take a family of geodesics through P given by the equation

$$n = \Phi(s, a) \tag{16.1}$$

where the parameter is chosen so that

$$a = \Phi_s(s_0, a). \tag{16.2}$$

Then for the geodesic C, $a = 0$ (cf. § 14). Let P_r with coordinates $(s_r, 0)$ be the rth conjugate point to P on C (cf. § 8). Let P_r' with coordinates (s_r', n') be the rth conjugate point to P on any geodesic a' of system (16.1). The function Φ of (16.1) will be supposed analytic in the domain

$$|a| \leqq d \qquad d > 0 \tag{16.3}$$

$$s_0 - e \leqq s \leqq s_r + e \qquad e > 0 \tag{16.4}$$

where d is taken small enough so that $s = s_r'$ satisfies (16.4) when $a = a'$ satisfies (16.3).

In the last section it was found that the locus of the points P_r' could be written in the form

$$s_r = f_1(a), \qquad n = f_2(a) \tag{16.5}$$

where f_1 and f_2 are analytic functions of a in the interval (16.3). In this section the possible forms that this envelope may take in the neighborhood of P_r will be discussed. Lindeberg* has investigated this problem for the

* *Mathematische Annalen*, Vol. 59 (1904), p. 321.

envelope in the neighborhood of the first conjugate point, and his results are easily extended to apply to the rth envelope. Let

$$M = (\partial^m/\partial a^m)\Phi(s_r, 0) \tag{16.6}$$

be the lowest order derivative of Φ with respect to a which does not vanish at P_r. It is found convenient to consider the possible forms which the rth envelope takes under five cases:

I. m is even and M negative.
II. m is even and M positive.
III. m is odd and M positive.
IV. m is odd and M negative.
V. M does not exist.

Case I. m is even and M negative. One denotes as the positive side of C near P_r the points neighboring C for which $n > 0$.

In the notation of this article equation (5) p. 324 of Lindeberg's article when applied to the rth envelope becomes *

$$n' = \Phi(s_r', a') - \Phi(s_r, 0) = (M/m - 2M)_{a'^m} + R_1(a') \tag{16.7}$$

where, as defined above, s_r' is the s coordinate of a point on the rth envelope at which a curve $a = a'$ of family (16.1) touches it, and $R_1(a')$ denotes a power series in a' all of whose terms are of higher power than the mth. The series $R_1(a')$ is known to converge for values of a' in the interval (16.3). It follows from (16.7) that *in Case I the envelope lies entirely on the positive side of C in the neighborhood of P_r.*

For the rth envelope the relation which Lindeberg writes at the top of page 324 becomes under the notation of this article

$$s_r' - s_r = (-1)^{r-1}C_1 M a^{m-1} + R_2(a') \tag{16.8}$$

where C_1 is a positive constant and $R_2(a')$ is a power series in a' all of whose terms are of higher power than the $m-1$th. $R_2(a')$ is known to converge in the interval (16.3). One replaces Lindeberg's $1/L^2$ by $(-1)^{r-1}C_1$ since it is clear from his definition of L^2 that its sign is positive or negative according as r is odd or even (cf. bottom of p. 322, Lindeberg, *loc. cit.*). From this formula it follows that *a neighboring geodesic to C of the family* (16.1) *through P determined by a' touches the envelope at a point P_r' for which*

* There is an error in (3) p. 323 of Lindeberg's article. It should read $2N$ instead of N. This accounts for the replacing of $-N$ in Lindeberg's formula (5) by $-2M$.

$s_r' - s_r$ is less than or greater than zero according as a' has the same sign as $(-1)^{r-1}$ or the opposite sign.

The formula of page 327 of Lindeberg's article becomes in the notation of this article

$$\bar{s} - s_r = (-1)^{r-1} C_2 M a'^{m-1}/m + R_3(a') \tag{16.9}$$

where C_2 is a constant greater than zero, $R_3(a')$ is of the same nature as $R_2(a')$ in (16.8), and where $\bar{s}$ gives the s coordinate of the intersection of the geodesic a' with the curve C in the neighborhood of P_r. This formula shows that *the neighboring geodesics to C through P for which a' has the same sign as $(-1)^{r-1}$ intersect C in the neighborhood of P_r at points between P and P_r while neighboring geodesics for which a' has the opposite sign intersect C on the far side of P_r from P.*

Case II. m is even and M positive. Here *the envelope near P_r is found to lie entirely on the negative side of C.* From (16.8) it is seen that *$s' - s_r$ has the same sign as $a'(-1)^{r-1}$ for neighboring geodesics to C through P.* Finally from (16.9) it is found that *neighboring geodesics to C through P for which $a'(-1)^{r-1} > 0$ intersect C in the neighborhood of P_r on the far side of P_r from P, and those for which $a'(-1)^{r-1} < 0$ intersect C between P and P_r near P_r.*

Case III. m is odd and M positive. Here it follows from formulae (16.7) and (16.8) that *the envelope has a cusp at P_r for which $s - s_r$ has the same sign as $(-1)^{r-1}$,* and that *a neighboring geodesic to C determined by a' in* (16.1) *touches the envelope at a point P_r' such that n' has the opposite sign to $a'(-1)^{r-1}$.* From (16.9) it follows that *neighboring geodesics to C intersect C near P_r at points on the farther or the nearer side of P^r from P according as $(-1)^{r-1}$ is positive or negative.*

Case IV. m is odd and M negative. Substitute $(-1)^r$ for $(-1)^{r-1}$ in the italicized statements under Case III and the results for Case IV are obtained.

Case V. m does not exist. Here *the equations* (16.5) *of the envelope give one point only,* and there is a system of geodesics through P and P_r.

17. *The rth envelope of geodesics of Classes IV-V through a point on the outside equator.* It has been seen that at a point on the outside equator the asymptotic geodesics cross this equator at an angle $\pm\chi_1$ (cf. § 7), whose cosine is given by the formula $\cos\chi_1 = r_1/r_2$. Geodesics of Class IV cross the outside equator at an angle χ, such that $\cos\chi = \alpha/r_2$, where α lies in the

interval $r_1 < a < r_2$. Recall that Class V consists of just one geodesic; namely, the outside equator, and on this geodesic $\chi = 0$. If there be given a point P on the outside equator, and any value of χ in the interval

$$-\chi_1 < \chi < +\chi_1, \tag{17.1}$$

the geodesic G through P whose positive tangent makes such an angle χ with the positive tangent to the outside equator, will be a geodesic of Class IV or V. Furthermore all geodesics of Classes IV-V through P will be determined by these values of χ. Let the coordinates (u, v) of P be $(u_0, 0)$. All the geodesics through $(u_0, 0)$ except $u = u_0$ can be represented in the form

$$v = \psi(u, b), \qquad b = \psi_u(u_0, b), \tag{17.2}$$

where $\psi(u, b)$ is analytic in u and b for all values of u and b. In Differential Geometry * there is the relation

$$\tan\chi = (G/E)^{\frac{1}{2}} dv/du = (1/r)\, dv/du.$$

Thus in particular at $(u_0, 0)$, at which point $r = r_2$,

$$b = \psi_u(u_0, b) = r_2 \tan\chi. \tag{17.3}$$

Corresponding to the varying of χ on the interval (17.1), b by virtue of the preceding relation will vary on an interval

$$-b_1 < b < +b_1 \tag{17.4}$$

where

$$b_1 = r_2 \tan\chi_1.$$

Now for the investigation of the rth envelope of the system of geodesics given by (17.2) for b in the interval (17.4). It is known from Theorem 10 (cf. § 15) that this rth envelope is given by equations of the form

$$u_r = U_r(b), \qquad v_r = V_r(b) \tag{17.5}$$

where U_r and V_r are analytic functions of b for b in (17.4). In order to investigate this envelope consider any geodesic C of family (17.2)-(17.4) and take normal geodesic coordinates with C as the "central geodesic." In the neighborhood of this geodesic C, the geodesics can be given in the form

$$n = \Phi(s, a) \tag{17.6}$$

$$a = \Phi_s(s_0, a) \tag{17.7}$$

where $n = 0$, or $a = 0$ gives the geodesic C and $(s_0, 0)$ is the point P.

In the last section it was found that in investigating the envelope of

* Eisenhart, *loc. cit.*, p. 26 (24.)

the geodesics given by (17. 6) that the lowest order derivative $(\partial^m/\partial a^m)\Phi(s, 0)$ which does not vanish on that envelope plays a very important rôle. Preparatory to proving a theorem which applies the theory to the envelopes of geodesics of Class IV two lemmas will be proved. Take the "central geodesic" C of system (17. 6)-(17. 7) as a geodesic for which $b \neq 0$ when represented in the form (17. 2). Let $\bar{P}_1$ denote the first intersection of C with the outside equator following P on C in the sense of increasing u. Let s be so measured on C that $s = 0$ on C at $\bar{P}_1$ and so that the sense of increasing s is the sense of increasing u. Recall from the general theory (B. § 40) that for the analytic, positive, definite problem of the Calculus of Variations the relation between the surface coordinates (u, v) and the surface coordinates (s, n) is

$$(17.8) \qquad u = u(s, n), \qquad v = v(s, n), \qquad D(u, v)/D(s, n) \neq 0,$$

where u and v are analytic functions of s and n for points in the neighborhood of a finite segment of C.

LEMMA 1. *On C, $(\partial/\partial n)K(s, n)$ is less than or greater than zero according as the v coordinate of the point $(s, 0)$ is greater or less than zero.*

Let $\bar{K}(u, v)$ be the surface curvature in terms of u and v. We can then write at the point given by (s, n) and (u, v)

$$\frac{\partial K}{\partial n} = \frac{\partial \bar{K}}{\partial v} \cdot \frac{\partial v}{\partial n} + \frac{\partial \bar{K}}{\partial u} \cdot \frac{\partial u}{\partial n}.$$

Denote by $\chi(s)$ the angle which the geodesic C makes with a parallel at a point $s = s$ on C [cf. (4. 6)]. Then since $\partial \bar{K}/\partial u$ is zero at $(s, n) = (s, 0)$,

$$(17.9) \qquad \frac{\partial K}{\partial n} = \frac{\partial \bar{K}}{\partial v} \cos \chi(s).$$

But it is known that $\partial u/\partial s$ is never zero on a geodesic of Class IV (cf. 4. 5), so $\cos \chi(s)$ will always be positive. For $v > 0$, $\partial \bar{K}/\partial v$ is negative and for $v < 0$, $\partial \bar{K}/\partial v$ is positive. Thus the lemma follows.

Let the coordinates (s, n) of the rth conjugate point

$$P_r, \ (r = 1, 2, 3, \cdots, m)$$

to P on C be $(s_r, 0)$. Let M and M_1 be two successive points on C at which extrema of v occur and which include the point $\bar{P}_1$ between them. Then from Theorem 8 it is seen that *P_1 will lie between M and its conjugate M_1 on C.* The second lemma concerns a solution $\bar{w}(s)$ of the Jacobi equation,

$$(17.10) \qquad w''(s) + K(s, 0)w(s) = 0,$$

which is zero at $s = s_0$ and which has first derivative equal to 1 there.

LEMMA 2. *If on the geodesic C emanating from P on $v = 0$ s be measured from $\bar{P}_1$, the first intersection of C with $v = 0$ following P, and if s_1 denote the s coordinate of the first conjugate point P_1 to P, then $\bar{w}(s)$, the solution of the Jacobi differential equation set up for C, vanishing at P, with a derivative equal to unity at P, satisfies the relation*

$$\bar{w}(-s) > \bar{w}(s) \tag{17.11}$$

for s in the interval

$$0 < s \leqq |s_1|. \tag{17.12}$$

In order to prove this lemma a function $w_1(s)$ is defined as follows:

$$w_1(s) = \bar{w}(-s). \tag{17.13}$$

It is then noted that

$$w_1''(s) = \bar{w}''(-s) \tag{17.14}$$

and on account of the symmetry of the geodesic C and the surface with respect to the equatorial plane that

$$K(s, 0) = K(-s, 0). \tag{17.15}$$

Accordingly $w_1(s)$ will be a solution of the Jacobi equation (17.10), for if one sets $s = -s$ in (17.10) and employs (17.13), (17.14) and (17.15) one obtains

$$w_1''(s) + K(s, 0) w_1(s) = 0.$$

If one sets

$$W(s) = \bar{w}(s) - w_1(s) = \bar{w}(s) - \bar{w}(-s) \tag{17.16}$$

and proves that $W(s) < 0$ on the interval (17.12), Lemma 2 will be demonstrated. Now $W(s)$ is a solution of (17.10) and, by virtue of (17.13), $W(0) = 0$. From (17.16) and (17.13)

$$W(s_1) = -\bar{w}(-s_1) \tag{17.17}$$

$$W(-s_1) = \bar{w}(-s_1). \tag{17.18}$$

Using (17.17) if $s_1 > 0$ and using (17.18) if $s_1 < 0$ one obtains

$$W(|s_1|) < 0. \tag{17.19}$$

Since $W(s)$ vanishes at $s = 0$ and M and M_1 are conjugate points (cf. Theorem 8), $W(s)$ can vanish at no other point between M and M_1. But $W(s)$ is negative at $s = |s_1|$. $W(s)$ must accordingly be negative on the whole interval (17.12), and Lemma 2 is proved.

Consider the following theorem.

THEOREM 11. *On a geodesic of Class IV through a point P on the outside equator and for which $b > 0$, $\Phi_{aa}(s_r, 0)$ is greater or less than zero at the rth conjugate point $s = s_r$ according as r is odd or even.. If $b < 0$, the sign of $\Phi_{aa}(s_r, 0)$ is reversed in the above statement.*

From formula (14.4)

$$\Phi_{aa}(s_r, 0) = (-1)^r A \int_{s_0}^{s_r} \bar{w}^3(t)\,(\partial K/\partial n)\,(t, 0)\,dt. \tag{17.20}$$

Now A is a positive constant. Take the central geodesic C as a geodesic of Class IV for which $b > 0$. Then the first part of Theorem 11 will be demonstrated if one shows that the integral

$$\int_{s_0}^{s_r} \bar{w}^3(t)\,(\partial K/\partial n(t, 0)\,dt \tag{17.21}$$

is negative. Divide the proof into three principal headings; I when $r = 1$, II when $r = 2$, III when $r > 2$.

I. $r = 1$.

Consider the two cases: Case A, when P_1 lies on the outside equator or on the side of the equatorial plane for which $v > 0$; Case B, when P_1 lies on the side of the equatorial plane for which $v < 0$.

CASE A: *The first conjugate point P_1 is at $\bar{P}_1$ or between M and P_1.*

It appears from Lemma 1 that in this case $(\partial K/\partial n)\,(s, 0)$ is negative between P and P_1. But $\bar{w}(t)$ is positive in this interval. Thus when $r = 1$ for Case A the integral (17.21) is negative.

CASE B: *The first conjugate point P_1 is between P_1 and M_1 (s is measured from $\bar{P}_1$).*

On account of symmetry

$$K(-s, -n) = K(s, n).$$

Hence

$$(\partial K/\partial n)\,(-s, 0) = -(\partial K/\partial n)\,(s, 0) \tag{17.22}$$

for all values of s. Write the integral (17.21) for $r = 1$ as

$$\int_{s_0}^{-s_1} \bar{w}^3(t)\,(\partial K/\partial n)\,(t, 0)\,dt + \int_{-s_1}^{0} \bar{w}^3(t)\,(\partial K/\partial n)\,(t, 0)\,dt + \int_{0}^{s_1} \bar{w}^3(t)\,(\partial K/\partial n)\,(t, 0)\,dt,$$

where s_1 is the value of s at P_1 and s_0 its value at P. The first of these integrals as in Case A is negative. In the third integral, from Lemma 2, $w(t) < w(-t)$ for t in the interval $0 < t \leqq s_1$. From formula (17.22)

and Lemma 2 it is found that the third integral is positive and is less in absolute value than the second integral, which is negative. Thus the sum of the three integrals is negative and accordingly integral (17.21) is negative for the case under consideration.

II. $r = 2$.

Let $\bar{P}_2$ denote the second intersection of the geodesic C with the outside equator following P in the sense of increasing u, and let M_2 denote the next point on C following M_1 at which v has an extremum. Write the integral (17.21) for $r = 2$ in the form

$$(17.23) \qquad \int_{s_0}^{|s_1|} \bar{w}^3(t) \frac{\partial K}{\partial n}(t,0)\,dt + \int_{|s_1|}^{s_2} \bar{w}^3(t) \frac{\partial K}{\partial n}(t,0)\,dt.$$

If $s_1 \geqq 0$ from the discussion in I Case B it is known that the first integral is negative. If, however, $s_1 < 0$ one can apply Lemma 2 and the fact that $w(t)$ is negative on the interval $0 < t < |s_1|$ to show that the first integral is negative just as was done in I Case B. In considering the second integral move the origin of measurement of s to P_2 and as before let s_0 be the s coordinate of P and s_r the rth conjugate point $(r = 1, 2, \cdots, m)$. Consider two cases:

$$\text{Case A, } s_2 \leqq 0. \qquad \text{Case B, } s_2 > 0.$$

In terms of the arc length measured from $\bar{P}_2$ let $\bar{s}_1$ denote the value of s at the point whose s coordinate was $|s_1|$ when s was measured from $\bar{P}_1$. Thus one can write the second integral of (17.23) as

$$(17.24) \qquad \int_{\bar{s}_1}^{s_2} \bar{w}^3(t) \frac{\partial K}{\partial n}(t,0)\,dt.$$

CASE A: $s_2 \leqq 0$.

Here $\bar{w}(t)$ is negative for this integral except at s_2 and perhaps $\bar{s}_1$. Also $\partial K/\partial n(t,0)$ is positive for the integral except possibly at s_2 and $\bar{s}_1$. Thus the integral is negative.

CASE B: $s_2 > 0$.

Write the integral (17.24) as

$$\int_{\bar{s}_1}^{-s_2} \bar{w}^3(t) \frac{\partial K}{\partial n}(t,0)\,dt + \int_{-s_2}^{0} \bar{w}^3(t) \frac{\partial K}{\partial n}(t,0)\,dt + \int_{0}^{s_2} \bar{w}^3(t) \frac{\partial K}{\partial n}(t,0)\,dt.$$

One treats these integrals as the three integrals of I Case B were treated. In the first integral $\bar{w}(t) < 0$ and $\partial K/\partial n > 0$ except possibly at the end values, and the integral is thus negative. A review of the proof of Lemma 2 will show that in II Case B $w(-s) < w(s)$ for the interval $0 < s \leqq |s_2|$. Accordingly the second integral above will be in absolute value larger than the third integral which is positive. The sum of all three will thus be negative.

III. $r > 2$.

Take $s = 0$ at the rth intersection $\bar{P}_r$ of C with the outside equator and consider the integral

$$\int_{\bar{s}_{r-1}}^{s_r} \bar{w}^3(t)\,(\partial K/\partial n)\,(t, 0)\,dt \tag{17.25}$$

where $\bar{s}_{r-1}$ is the s coordinate in terms of arc length measured from $\bar{P}_r$, of the point whose s coordinate in terms of arc length measured from $\bar{P}_{r-1}$ was $|s_{r-1}|$. If r is odd this integral can be proved negative by treating as in I; and if r is even it can be treated as the second integral of (17.23) was treated in II. If s_r is negative the integral

$$\int_{s_r}^{|s_r|} \bar{w}^3(t)\,(\partial K/\partial n)\,(t, 0)\,dt \tag{17.26}$$

also has to be considered, but this integral offers no new difficulty and can be shown to be negative by using Lemma 2 as in I if r is odd, and as in II with a reversal of sign, if r is even. Thus combining (17.25) and (17.26) it follows from I and II by mathematical induction that the integral (17.21) for $r > 2$ is negative when $b > 0$. If $b < 0$, from the symmetry of the situation as evidenced in formula (17.22), it is seen that the integral will be positive. Theorem 11 is thus demonstrated.

One can now prove the interesting theorem:

THEOREM 12. *The rth envelope to geodesics of Classes IV-V through a point P on the outside equator is given by equations* (17.5) *where* U_r *and* V_r *are analytic for* b *in the interval* (17.3). *It is not a point and is symmetric with respect to the equatorial plane of the surface. If* r *is odd; when* $b > 0$ *the envelope is tangent to the geodesics as in Case II,* § 16, *and when* $b < 0$ *it is tangent to the geodesics as in Case I. If* r *is even; when* $b > 0$ *Case I applies, and when* $b < 0$ *Case II applies.*

That such an envelope exists follows from Theorem 10. That it is not a point follows at once from Theorem 11 and § 16. Because of the symmetry of the surface and the geodesics issuing from P, the envelope will be symmetrical in the equatorial plane. The facts presented in the last two statements of the theorem are easily deduced from Theorem 11 and the "Im Kleinem" envelope theory of § 16.

23. *Further theorems concerning the rth envelope of geodesics through a point on the outside equator.* The following theorems can be proved by methods similar to those used in the foregoing sections. They are stated in the order in which their proofs would naturally be given.

THEOREM 13. *The rth envelope to geodesics of Classes IV-V through a point P on the outside equator has a cusp on the outside equator at the point $(u_r, 0)$ for which $u \geqq u_r$. The envelope has no other singularities. The vertex of the cusp divides the envelope into two branches on either of which v is an analytic function of u. On the branch of the envelope for which $v > 0$, $v' > 0$; and on the branch for which $v < 0$, $v' < 0$.*

THEOREM 14.

(a) *On a geodesic of Class IV the maximum at $r = \alpha$ of v increases continuously as b increases from 0 to b_1, where b_1 is the positive value of b corresponding to one of the asymptotic geodesics through P.*

(b) *The coordinate u of the first point of intersection with the outside equator of a geodesic of Class IV through a fixed point P on the outside equator increases continuously and without limit as b increases in the open interval from 0 to b_1.*

THEOREM 15. *The two branches of the rth envelope of geodesics of Class IV through a point $u = u_0$ on the outside equator for which $v > 0$ and $v < 0$ become asymptotic to the inside equator on the side for which $v > 0$ and $v < 0$ respectively.*

THEOREM 16. *For $v = \bar{v}$ let $\bar{u}_r$ and $\bar{u}_{r+1}$ be the corresponding u coordinates of points on the rth and $r + 1$th envelopes respectively. The difference $\bar{u}_{r+1} - \bar{u}_r$ increases continuously and without limit as v increases from 0 to the value of v on the inside equator.*

COROLLARY 1. *The rth conjugate point P_r to the point P on the outside equator for a geodesic of Class IV recedes from the outside equator as r is taken larger and larger.*

COROLLARY 2. *No two of the rth envelopes have a point in common.*

COROLLARY 3. *Let b_m denote the geodesic of Class IV which has its extreme values of v on the parallels $v = \pm \bar{v}$, and let $d(b_m)$ denote the difference of the u coordinates of successive extrema on b_m. As r becomes infinite, for v constant, $\bar{u}_{r+1} - \bar{u}_r$* (see Th. 16) *decreases monotonically but always remains greater than $d(b_m)$.*

Somewhat similar theorems can be demonstrated concerning the envelopes of a system of geodesics of Class IV through a point not on the outside equator.

CORNELL UNIVERSITY.

Involutorial Transformations Belonging to a Linear Complex.

By HANNIBAL ALBERT DAVIS.

Introduction. Involutorial space transformations belonging to a linear complex have been studied in considerable detail, especially by Montesano * and Pieri.† The methods employed by them are entirely synthetic. In the present paper the equations of the transformation belonging to a non-special linear complex are derived in the general case and in some of the particular cases. A number of new transformations are discussed. It has seemed advisable to include some of the results already known. The results in sections 1, 4-6, 8, 10-12, 16-20 are due wholly or in part to Montesano, and those in Section 21 to Pieri.

1. *Transformations I Belonging to a Non-Special Linear Complex.*

1. *General Case.* Let the pairs of conjugate points P_1, P_2 in an involutorial space transformation I lie on the rays of a non-special linear complex Γ, a single pair on an arbitrary ray. The self-conjugate surface F in a linear congruence Q_1 of Γ, locus of the pairs of conjugate points situated on the lines of the congruence, contains simply the directrices, p, q of Q_1, and passes through the pair of conjugate points on each line of Q_1, hence is of order 4. A plane through p (or q) cuts F_4 in a residual cubic Δ_3 of genus 1, the self-conjugate curve in the Γ-pencil lying in the plane. Two cubics Δ_3 associated with two Γ-pencils having a line in common meet in the pair of conjugate points on the common line, hence the self-conjugate curve of a Γ-regulus R is a Δ_6 of genus 3 lying on the quadric containing R and having the lines of R as bisecants.

Since the base of a pencil of linear congruences is a regulus R, the pencil $|F_4|$ of associated self-conjugate surfaces has for base, in addition to the fundamental curve of the transformation, a Δ_6 of genus 3; hence the fundamental curve is a C_{10} of genus 11. Any plane π through an arbitrary line

* "Su le trasformazioni involutorie dello spazio chi determinano un complesso lineare di rette," *Roma Accademia dei Lincei Rendiconti*, Ser. 4, Vol. 4 (1888), pp. 207-215, 277-285.

† "Sulle trasformazioni birazionali dello spazio inerenti a un complesso lineare speciale," *Circolo Matematico di Palermo*, Vol. 6 (1892), pp. 240-244.

p not meeting C_{10} contains a Δ_3 associated with the Γ-pencil in π. The curves p and Δ_3 form the complete section of a surface F_4 by π, hence the fundamental curve C_{10} meets any plane in 10 points situated on a cubic curve Δ_3.

Since the image in I of a point P on C_{10} is a Δ_3 with a double point at P, the C_{10} is triple on the surfaces conjugate to the planes of space and the transformation is of order 11. The surfaces conjugate to the planes of space are Φ_{11}: $C^3{}_{10}\ 20a_i$, where the a_i are parasite lines, quadrisecants of C_{10}.

The surface of invariant points is K_8: $C_{10}{}^2\ 20a_i$.

The jacobian of the transformation is the image surface J_{40}: $C_{10}{}^{11}20a_i{}^4$ of the fundamental C_{10}.

Since the conjugate in I_{11} of a point P on C_{10} is a Δ_3: P^2, this Δ_3 meets C_{10} in 11 points. In general, the conjugate J of a fundamental curve C contains C to multiplicity equal to the number of points of intersection of C with the image curve of a point on it.

2. *The Equations of the Transformation.* There are $\infty^4\ |\ F_4\ |$, one for each linear congruence in Γ. Each F_4 of the web through a point P_1 passes also through a second point P_2 conjugate to P_1 in the I_{11}. But a single pencil $|\ F_4\ |$ of these surfaces is sufficient to determine the transformation.

The directrices of a pencil $|\ Q_1\ |$ of linear Γ-congruences form a regulus R', lying on the quadric H_2 with the regulus R, base of the pencil $|\ Q_1\ |$. Through an arbitrary point P_1 of space passes one surface F_4 of the pencil $|\ F_4\ |$ associated in I_{11} with the pencil $|\ Q_1|$. The unique line from P_1 meeting the directrices of the congruence Q_1 associated with F_4 meets the F_4 in one other point P_2, conjugate to P_1.

Denote by

(1) $x_1x_3 - x_2x_4 = 0$, (2) $(x_1/x_2)=(x_4/x_3)=k$, (3) $(x_1/x_4)=(x_2/x_3)=m$,

the quadric H_2, the lines of R, and the lines of R' respectively. An F_4 which intersects (1) in a pair of lines (3) and a residual Δ_6 is of the form

$$m^2f_1(k) + mf_2(k) + f_3(k) = 0,$$

where $f_i(k) = a_ik^4 + b_ik^3 + c_ik^2 + d_ik + e_i$, $(i = 1, 2, 3)$.
Two such surfaces are;

$$\begin{aligned}
F \equiv\ & a_1x_1{}^4 + b_1x_1{}^3x_2 + c_1x_1{}^2x_2{}^2 + d_1x_1x_2{}^3 + e_1x_2{}^4 \\
& a_2x_1{}^3x_4 + b_2x_1{}^2x_2x_4 + c_2x_1x_2{}^2x_4 + d_2x_1{}^2x_2x_3 + e_2x_2{}^3x_3 \\
& + a_3x_1{}^2x_4{}^2 + b_3x_1x_2x_4{}^2 + c_3x_2{}^2x_4{}^2 + d_3x_2{}^2x_3x_4 + e_3x_2{}^2x_3{}^2 = 0, \\
F' \equiv\ & a_1x_1{}^2x_4{}^2 + b_1x_1{}^2x_3x_4 + c_1x_1{}^2x_3{}^2 + d_1x_1x_2x_3{}^2 + e_1x_2{}^2x_3{}^2 \\
& + a_2x_1x_4{}^3 + b_2x_1x_3x_4{}^2 + c_2x_1x_3{}^2x_4 + d_2x_1x_3{}^3 + e_2x_2x_3{}^3 \\
& + a_3x_4{}^4 + b_3x_3x_4{}^3 + c_3x_3{}^2x_4{}^2 + d_3x_3{}^3x_4 + e_3x_3{}^4 = 0.
\end{aligned}$$

Each F_4 of the pencil

$$F - \rho F' = 0 \tag{4}$$

contains two variable lines of R' and the Δ_6. Through a point $P_1(y)$ of space passes one F_4 of (4),

$$F - p^2 F' = 0, \tag{5}$$

where $p^2 = F(y)/F'(y)$. Substituting F and F' in (5), we obtain

$$\begin{aligned}(6)\quad & (a_1x_1^2 + a_2x_1x_4 + a_3x_4^2)\ (x_1^2 - p^2x_4^2) + (b_1x_1^2 + b_2x_1x_4 + b_3x_4^2)\\ & (x_1x_2 - p^2x_3x_4) + (c_1x_1^2 + c_2x_1x_4 + c_3x_4^2 + d_1x_1x_2 + d_2x_1x_3\\ & + d_3x_3x_4 + e_1x_2^2 + e_2x_2x_3 + e_3x_3^2)(x_2^2 - p^2x_3^2) = 0.\end{aligned}$$

This surface contains the two lines of R',

$$g_1 \equiv x_1 - px_4 = 0,\ x_2 - px_3 = 0;\ g_2 \equiv x_1 + px_2 = 0,\ x_2 + px_3 = 0.$$

The line r through $P_4(y)$ meeting g_1, g_2 meets them in $A(\alpha)$ and $B(\beta)$ respectively, where

$$\begin{aligned}&\alpha_1 = p(y_1 + py_4),\ \alpha_2 = p(y_2 + py_3),\ \alpha_3 = y_2 + py_3,\ \alpha_4 = y_1 + py_4,\\ &\beta_1 = -p(y_1 - py_4),\ \beta_2 = -p(y_2 - py_3),\ \beta_3 = y_2 - py_3,\ \beta_4 = y_1 - py_4.\end{aligned}$$

Any point on r has coordinates;

$$\begin{aligned}\sigma x_1 &= \lambda\alpha_1 + \mu\beta_1 = \tau(y_1 + p^2ny_4),\\ \sigma x_2 &= \lambda\alpha_2 + \mu\beta_2 = \tau(y_2 + p^2ny_3),\\ \sigma x_3 &= \lambda\alpha_3 + \mu\beta_3 = \tau(y_3 + ny_2),\\ \sigma x_4 &= \lambda\alpha_4 + \mu\beta_4 = \tau(y_4 + ny_1),\end{aligned} \tag{7}$$

where $n = (\lambda + \mu)/p(\lambda - \mu)$. Substituting (7) in (6) and recalling that P, A, and B lie on (6), we get

$$\begin{aligned}(8)\quad & [a_1p^2y_4(2y_1 + p^2ny_4) + a_2(y_1^2 + p^2ny_1y_4 + p^2y_4^2)\\ & + a_3y_1(ny_1 + 2y_4)]\ [F'y_1^2 - Fy_4^2] + [b_1p^2y_4(2y_1 + p^2ny_4)\\ & + b_2(y_1^2 + p^2ny_1y_4 + p^2y_4^2) + b_3y_1(ny_1 + 2y_4)]\ [F'y_1y_2 - Fy_3y_4]\\ & + [c_1p^2y_4(2y_1 + p^2ny_4) + c_2(y_1^2 + p^2ny_1y_4 + p^2y_4^2)\\ & + c_3y_1(ny_1 + 2y_4) + e_1p^2y_3(2y_2 + p^2ny_3) + e_2(y_2^2 + p^2ny_2y_3 + p^2y_3^2)\\ & + e_3y_2(ny_2 + 2y_3) + d_1p^2(y_1y_3 + y_2y_4 + p^2ny_3y_4)\\ & + d_2(y_1y_2 + p^2y_3y_4 + p^2ny_2y_4) + d_3(y_1y_3 + y_2y_4\\ & \qquad + ny_1y_2)]\ [F'y_2^2 - Fy_3^2] = 0,\end{aligned}$$

where F and F' mean $F(y)$ and $F'(y)$. One notes that

$$\begin{aligned}&F'y_1^2 - Fy_4^2 = [y_1y_3 - y_2y_4]\ [by_1y_4 + (c + d + e)(y_1y_3 + y_2y_4)],\\ &F'y_1y_2 - Fy_3y_4 = [y_1y_3 - y_2y_4]\ [-ay_1y_4 + (c + d + e)y_2y_3],\\ &F'y_2^2 - Fy_3^2 = [y_1y_3 - y_2y_4]\ [-a(y_1y_3 + y_2y_4) - by_2y_3],\end{aligned}$$

where $a = a_1y_1^2 + a_2y_1y_4 + a_3y_4^2$, $b = b_1y_1^2 + b_2y_1y_4 + b_3y_4^2$,
$c = c_1y_1^2 + c_2y_1y_4 + c_3y_4^2$, $d = d_1y_1y_2 + d_2y_1y_3 + d_3y_3y_4$,
$e = e_1y_2^2 + e_2y_2y_3 + e_3y_3^2$.

Equation (8) is now of the form

$$Pn + QF' = 0. \tag{9}$$

If we set

$$A = by_1y_4 + (c + d + e)(y_1y_3 + y_2y_4),$$
$$B = -ay_1y_4 + (c + d + e)y_2y_3,$$
$$C = -a(y_1y_3 + y_2y_4) - by_2y_3,$$

the coefficients in (9) become

$$P = [a_1F^2y_4^2 + a_2FF'y_1y_4 + a_3F'^2y_1^2]A + [b_1F_2y_4^2 + b_2FF'y_1y_4 + b_3F'^2y_1^2]B + [c_1F^2y_4^2 + c_2FF'y_1y_4 + c_3F'^2y_1^2 + d_1F^2y_3y_4 + d_2FF'y_2y_4 + d_3F'^2y_1y_2 + e_1F^2y_3^2 + e_2FF'y_2y_3 + e_3F'^2y_2^2]C,$$

$$Q = [2(a_1F + a_3F')y_1y_4 + a_2(F'y_1^2 + Fy_4^2)]A + [2(b_1F + b_3F')y_1y_4 + b_2(F'y_1^2 + Fy_4^2)]B + [2(c_1F + c_3F')y_1y_4 + c_2(F'y_1^2 + Fy_4^2) + (d_1F + d_3F')(y_1y_3 + y_2y_4) + d_2(F'y_1y_2 + Fy_3y_4) + 2(e_1F + e_3F')y_2y_3 + e_2(F'y_2^2 + Fy_3^2)]C.$$

Since $n = -F'Q/P$, the transformation is obtained from (7).

$$x_1 = Py_1 - FQy_4,\ x_2 = Py_2 - FQy_3,\ x_3 = Py_3 - F'Qy_2,\ x_4 = Py_4 - F'Qy_1. \tag{10}$$

This is apparently of order 15, but there is a common factor $(y_1y_3 - y_2y_4)^2$. The factor $(y_1y_3 - y_2y_4)$ is contained once in both P and Q, and once more after the resulting expressions are combined in (10).

3. *Representations of I_{11} on a Double Space.* If the Plücker line coördinates are used, the complex to which I_{11} belongs may be written

$$\Gamma \equiv p_{12} - p_{34} = 0.$$

The lines of Γ may then be represented by the points of a quadric variety in 4-space by means of the transformation

$$p_{12} = p_{34} = \rho x_1,\ p_{13} = \rho x_2,\ p_{14} = \rho x_3,\ p_{42} = \rho x_4,\ p_{23} = \rho x_5, \tag{11}$$

$$x_1^2 + x_2x_4 + x_3x_5 = 0. \tag{12}$$

The projection of the points of this quadric from (0, 0, 1, 0, 0) into an ordi-

nary space $\sum_3'$ gives the Lie-Noether representation of the lines of Γ on the points of $\sum_3'$,*

$$(13) \quad p_{12} = p_{34} = \sigma x_1' x_3', \; p_{13} = \sigma x_2' x_3', \; p_{14} = -\sigma(x_1'^2 + x_2' x_4'),$$
$$p_{42} = \sigma x_3' x_4', \; p_{23} = \sigma x_3'^2,$$

$$(13') \quad x_1' = \tau p_{12} = \tau p_{34}, \; x_2' = \tau p_{13}, \; x_3' = \tau p_{23}, \; x_4' = \tau p_{42},$$
$$x_1^2 + x_2 x_4 + x_3 x_5 = 0.$$

This (1, 1) representation of the lines of Γ upon the points of $\sum_3'$ defines a (1, 2) point transformation between the space $\sum_3$ of Γ and the space $\sum_3'$; the pair of conjugate points P_1, P_2 situated on a line of Γ corresponding to the image point P' in $\sum_3'$ of the line.

The intersection with Γ of the complex

$$(14) \quad a_1 p_{12} + a_2 p_{13} + a_3 p_{14} + a_4 p_{42} + a_5 p_{23} = 0$$

is a linear congruence, whose image in $\sum_3'$ is

$$(a_1 x_1' + a_2 x_2' + a_5 x_3' + a_4 x_4') x_3' - a_3(x_1'^2 + x_2' x_4') = 0.$$

Hence, $\infty^4 \mid F_4 \mid \sim \infty^4 \mid H_2' \mid$: γ_2' where $\gamma_2' \equiv x_3' = 0$, $x_1'^2 + x_2' x_4' = 0$.

Associated with (0, 0, 1, 0, 0,) in $\sum_4$ is a line u of Γ which contains a pair of conjugate points U_1, U_2.

$$u = p_{14} = 1, \; p_{ik} = 0, \; (ik \neq 14).$$

From (13), $x_3' = 0$, $x_1'^2 + x_2' x_4' \neq 0$. Therefore u (or U_1, U_2) $\sim$ any point of the plane $\alpha' \equiv x_3' = 0$ not on the conic γ_2'.

Since the condition that the complex (14) contains u is $a_3 = 0$, it follows that $\infty^3 \mid F_4 \mid$: U_1, $U_2 \sim \infty^3 \mid H_2' \mid \equiv \infty^3 \mid \pi' \mid + \alpha'$, where the π' are the planes of $\sum_3'$

It is easy to see that; $\infty^6 \mid \Delta_6 \mid \sim \infty^6 \mid C_2' \mid$, where $[C_2', \gamma_2'] = 2$, $\infty^4 \mid \Delta_6 \mid$: U_1, $U_2 \sim \infty^4 \mid r' \mid$, where the r' are the lines of $\sum_3'$ not meeting γ_2'. The remainder of the composite C_2' is a line in α' meeting r'. $\infty^3 \mid \Delta_3 \mid \sim \infty^3 \mid \delta' \mid$, where the δ' are lines meeting γ_2'.

There are in a linear Γ-congruence Q_1, with distinct directrices p, q two series of Γ-pencils, one series for each directrix. The two series of Δ_3's belonging to the pencils correspond in $\sum'_3$ to the two sets of generators of the quadric H_2, image of the F_4 associated with Q_1. If p coincides with q and hence belongs to Γ, the congruence Q_1 contains a single series of Γ-pencils. The

* See e. g. Sturm, *Liniengeometrie*, Vol. 1, pp. 257-269.

image in Σ_3' of the F_4 is in this case a quadric cone through γ_2' whose vertix is the image of the pair of points P_1, P_2 on p, conjugate in the I_{11}.

Let P_1 in Σ_3 describe a line p, which together with its polar line q in Γ determines a linear Γ-congruence, hence an F_4. The image in Σ_3' of p must be a curve C_n' on the H_2' which is the image of the F_4. A plane through p cuts F_4 in a residual Δ_3, self-conjugate curve in the I_{11} of the Γ-pencil in the plane. The Δ_3 meets p in three points and q in one point. Similarly, the Δ_3 in a plane through q meets p in one point and q in three points. Hence, the image in Σ_3' of p (together with its conjugate c_{11} in I_{11}) is a C_4' of type [3, 1], while the image of q, polar of p in Γ, is a C_4' of type [1, 3] on the same quadric.

The image in Σ_3' of a line p of Γ is a cubic C_3' on the cone K_2', image of the linear Γ-congruence with p for directrix. The C_3' passes through the vertex of K_2' and cuts each of its generators in one other point.

A Γ-congruence Q_n of degree n corresponds in Σ_3' to a surface $F' : \gamma_2'^n$. Q_n contains n lines of each Γ-pencil and $2n$ lines of each Γ-regulus. The image of a Γ-pencil is a line δ' meeting γ_2', and the image of a Γ-regulus containing u is a line L' which does not meet γ_2'. Hence, $Q_n \sim F'_{2n} : \gamma_2'^n$, provided Q_n does not contain u.

In particular, as P_1 describes a plane π, the Γ-lines P_1P_2 form a congruence Q_n. An arbitrary plane α contains a Γ-pencil, hence a Δ_3, which cuts π in three points. The three lines of the Γ-pencil in α through these three points belong to Q_n, hence $n = 3$. Therefore a plane (together with its conjugate Φ_{11} in I_{11}) corresponds in Σ_3' to a surface $F_6' : \gamma_2'^3$, provided the plane contains neither U_1 nor U_2.

The surface of invariant points in I_{11} was seen to be $K_8 : C_{10}^2$.

$$[K_8, \Delta_6] = 48, \quad [K_8, \Delta_3] = 24, \quad [C_{10}, \Delta_6] = 20, \quad [C_{10}, \Delta_3] = 10.$$

Since $\Delta_3 \sim$ a line meeting γ_2', and $\Delta_6 : U_1, U_2 \sim$ a line not meeting γ_2', it follows that the image Σ_3' of K_8 is $L_8' : \gamma_2'^4$.

The image in Σ_3' of the fundamental curve C_{10} is a ruled surface $R'_{20} : \gamma_2'^{10}$.

A few details of the Lie-Noether representation will now be considered. The corresponding details of the (2, 1) point transformation are easily supplied.

The linear Γ-congruence (Γ, u) composed of the lines of Γ which meet u is the intersection of Γ with the special linear complex $p_{23} = 0$. From (13) it follows that $(\Gamma, u) \sim \alpha'^2$.

A point U on u has coördinates $(k, 0, 0, 1)$. A point in the polar plane

of U with respect to Γ is $(\alpha_1, \alpha_2, -k\alpha_2, \alpha_4)$. Hence, any line of the Γ-pencil with vertex U has coördinates: $p_{12} = k\alpha_2$, $p_{14} = k\alpha_4 - \alpha_1$, $p_{42} = \alpha_2$, $p_{13} = -k^2\alpha_2$, $p_{23} = 0$, $p_{34} = k\alpha_2$. Equations (13) give as the image in $\sum_3'$ the point $(k, -k^2, 0, 1)$ on γ_2' Hence, each line of a Γ-pencil meeting u corresponds to the same point on γ_2'.

The variable image of a linear Γ-congruence containing u was seen to be a plane π'. Such a congruence with distinct directrices p, q contains two series of pencils, which correspond to the two pencils of lines $[\pi' \gamma_2']$, whose vertices represent the two pencils with vertices on u. But if $p \equiv q$, the congruence contains a single series of pencils and a single pencil with vertex on u. The image plane π' is tangent to γ_2' at a point O', image of each line of the pencil on u.

An arbitrary Γ-pencil (A, α) contains a single line p meeting u in a point O. As A describes p, the image line δ' of (A, α) generates the pencil $O', \pi')$, where π', image of the linear Γ-congruence with directrix p, is tangent to γ_2' at O', image of p. As p describes the Γ-pencil on O, the pencil (O', π') describes the bundle O'. As the point O describes u, the point O' describes γ_2', and we have the pencils of Γ represented by the lines meeting γ_2' as before noted.

4. If the fundamental C_{10} becomes composite in such a way as to have a ruled surface S_μ of parasitic lines, quadrisecants of the composite C_{10} the surface S_μ is a part of each of the surfaces Φ, conjugates of planes in the I. Hence the transformation is of order $11 - \mu$. If the components of C_{10} are C_p, C_q, C_r, C_s of orders p, q, r, s, $p + q + r + s = 10$, and of multiplicities 0, 1, 2, 3, respectively, on S_μ, the images of planes are $\Phi_{11-\mu}$: $C_p^3\, C_q^2\, C_r\, a_1 \cdots a_k$. The parasitic lines a are quadrisecants of the composite C_{10} which are not generators of S_μ.

The surface of invariant points is $K_{8-\mu}$: $C_p^2\, C_q\, a_1 \cdots a_k$. The C_s is a singular invariant curve of the transformation.

The lines of Γ which meet C_p form a congruence Q_p of degree p. The self-conjugate surface F_{4p}: $C_p^{p+1}\, C_q^p\, C_r^p\, C_s^p$ in Q_p is composed of the image surface of J of C_p and the S_μ counted λ_1 times, where λ_1 is the number of points in which a generator of S_μ meets C_p. Hence the conjugate of C_p is

$$J_{4p-\lambda_1\mu}: C_p^{p+1}\, C_q^{p-\lambda_1}\, C_r^{p-2\lambda_1}\, C_s^{p-3\lambda_1}.$$

Similarly the conjugates of C_q, C_r and C_s are, respectively,

$$J_{4q-\lambda_2\mu}: C_p^q\, C_q^{q+1-\lambda_2}\, C_r^{q-2\lambda_2}\, C_s^{q-3\lambda_2},$$
$$J_{4r-\lambda_3\mu}: C_p^r\, C_q^{r-\lambda_3}\, C_r^{r+1-2\lambda_3}\, C_s^{r-3\lambda_3},$$
$$J_{4s-\lambda_4\mu}: C_p^s\, C_q^{s-\lambda_4}\, C_r^{s-2\lambda_4}\, C_s^{s+1-3\lambda_4},$$

Since $p+q+r+s=10$ and $\lambda_1+\lambda_2+\lambda_3+\lambda_4=4$, the sum of the orders of the surfaces J is $40-4\mu$; hence they form the complete jacobian of the transformation. The C_s being a singular invariant curve, $4s-\lambda_{4\mu}=0$. The multiplicity of each line a on J is equal to the number of points in which a meets that part of the fundamental curve associated with J.

The image in $\sum_3'$ of S_μ is in general a curve C_μ' which meets γ_2' in μ points. If the line u is a generator of S_μ, the image of S_μ is a $C_{\mu-1}'$ which meets γ_2' in $\mu-2$ points. In the former case $C_p \sim R_{2p}' : \gamma_2'^p$, $C_q \sim R_{2q}' : \gamma_2'^q$, $C_r \sim R_{2r}' : \gamma_2'^r$, $C_s \sim R_{2s}' : \gamma_2'^s$. The multiplicity of C_μ' on any R' is the number of points in which a generator of S_μ meets that part of the fundamental curve associated with the R'.

5. Suppose the surface S_μ is a pencil (A, α) of Γ. The self-conjugate surfaces in the net of linear Γ-congruences which contain the pencil (A, α) are, aside from the plane α, a net of cubic surfaces $|F_3|$. The self-conjugate curve of the Γ-pencil in a plane β through A is, aside from the parasitic line (α, β), a conic Δ_2 which, together with that part of the fundamental C_{10} not in α, forms the base of a pencil of the net $|F_3|$. Hence the C_{10} is composed of a C_7 of genus 5 through A and a γ_3 of genus 1 in α which meets the C_7 in the 6 points $[\alpha, C_7]$ other than A. The curve γ_3 is not the Δ_3 of α. The self-conjugate conic in any plane β through A meets C_7 in the 6 points $[\beta, C_7]$, other than A.

Through an arbitrary point P_1 of space passes a pencil of the net $|F_3|$, hence a Δ_2, self-conjugate in the Γ-pencil in a plane β through A. The line of Γ in β through P_1 meets Δ_2 in a second point P_2, conjugate to P_1 in the I_{10}.

The image in $\sum_3'$ of the net $|F_3| : C_7$ is the net $|H_2'| : \gamma_2'\delta'$, where δ' is the image of the pencil (A, α) in the Lie-Noether representation. The images of C_7 and γ_3 are, respectively, $R_{14}' : \gamma_2'^7\,\delta'$ and $R_6' : \gamma_2'^3\,\delta^3$. $[R_{14}', R_6'] = \gamma_2'^{21}\,\delta'^3\,6\delta_i'\,C_{33}'$, where the $6\delta_i'$ are the images of the 6 Δ_2 associated in the I_{10} with the 6 points $[C_7, \gamma_3]$, and the C_{33}' is the image of the self-conjugate curve on the ruled surface R_{33} of lines meeting both C_7 and γ_3. The R_{33} does not include the pencil (A, α) nor the Γ-pencils through the 6 points $[C_7, \gamma_3]$.

6. Suppose the surface S_μ is a regulus R. The self-conjugate surfaces in the pencil of linear Γ-congruences whose base is R break up into the quadric S_2 of R and a pencil of quadric surfaces $|H|$. The fundamental C_{10} is composed of the C_4 of genus 1, base of the pencil $|H|$, and a hyperelliptic γ_6 of genus 3 on S_2 which meets C_4 in 8 points and which has for quadrisecants the lines of R.

Let the lines of the regulus R' of S_2, different from R, be $x_1/x_4 =$

$x_2/x_3 = m$. The complex Γ is cut by a pencil $A + tB = 0$ of linear complexes in a pencil of linear congruences whose self-conjugate surfaces are composed of S_2 and a pencil $H_1 + tH_2 = 0$ of quadric surfaces. With each quadric of the pencil associate a pair of lines of R' by means of the relation $m^2 = t$. Through a point P_1 of space passes one quadric H of the pencil. The line from P_1 cutting the pair of lines of R' associated with H meets H in a second point P_2 conjugate to P_1 in the I_9 thus defined. The equations of the I_9 are easily obtained and are free from extraneous factors.

The image in $\sum_3'$ of the pencil $|H| : C_4$ is the pencil of quadrics $|H_2'| : \gamma_2'\ C_2'$, where C_2' is the image of the regulus R in the Lie-Noether representation. The images of C_4 and γ_6 are, respectively, $R_8' : \gamma_2'^4$ and $R_{12}' : \gamma'^6\ C'^2$. $[R_8' R_{12}'] = \gamma_2'^{24}\ 8\delta_i'\ C_{40}'$, where the $8\delta_i'$ are the images of the 8 Δ_2 associated in the I_9 with the 8 points $[C_4, \gamma_6]$; and the C_{40}' is the image of the self-conjugate curve on the ruled surface R_{40} of lines of Γ meeting both C_4 and γ_6 other than the 8 pencils through their points of intersection.

7. Suppose there are two pencils (A, α) and (B, β) of parasitic lines which do not have a line in common. The fundamental curves are the line $[\alpha, \beta] = k$, two conics C_2 in α and γ_2 in β, and a C_5 of genus 2. $[C_2, \gamma_2] = 0$, $[C_5, k] = 0$, $[C_5, C_2] = 4$, $[C_5, \gamma_2] = 4$.

The conjugates of planes in the I_9 are $\Phi_9 : C_5^3\ C_2^2\ \gamma_2^2\ k\ 8a_i\ 2b_i$. The a_i are bisecants of C_5 meeting C_2 and γ_2, and the b_i are the trisecants of C_5 meeting k. These lines b_i are the only trisecants of C_5 which belong to Γ.

The surface of invariant points is $K_6 : C_5^2\ C_2\ \gamma_2\ 8a_i\ 2b_i$.

The conjugates in I_9 of the fundamental curves k, C_2, γ_2 and C_5 are, respectively, $J_2 : C_5\ 2b_i$, $J_6 : C_5^2\ C_2\ \gamma_2^2\ 8a_i$, $J_6 : C_5^2\ C_2^2\ \gamma_2\ 8a_i$, and $J_{16} : k^3\ C_5^6\ C_2^4\ \gamma_2^4\ 8a_i^2\ 2b_i^3$. These surfaces form the jacobian of the transformation.

The pencils (A, α) and (B, β) are represented in $\sum_3'$ by the lines δ_1' and δ_2', respectively; and the images of the fundamental curves k, C_2, γ_2 and C_5 are, respectively, $R_2' : \gamma_2'\ \delta_1'\ \delta_2'$, $R_4' : \gamma_2'^2\ \delta_1'^2$, $R_4' : \gamma_2'^2\ \delta_2'^2$, and $R_{10}' : \gamma_2'^5$.

The surface $J_2 : C_5$, together with the planes α and β, forms the F_4 associated in the I with the linear Γ-congruence which contains the pencils (A, α) and (B, β). J_2 is represented in $\sum_3'$ by $R_2' = \gamma_2'\ \delta_1'\ \delta_2'$, image of k.

Since a composite regulus is made up of two pencils with a common line, the transformation discussed here is quite distinct from that of Section 6.

8. Suppose the surface of parasitic lines is an $S_3 : k^2$, where k is necessarily fundamental. A composite case is a regulus R and a pencil (A, α), the line k being part of the fundamental curve on each. The residual fundamental curves on R and α are, respectively, a γ_5 of genus 2 and a γ_2, each

meeting k in 2 points. Then γ_5 and γ_2 meet in 2 points. Therefore the fundamental curves are, besides the line k, a curve γ_7 of genus 3 on S_3 and a C_2 not on S_3. $[\gamma_7, C_2] = 6$. $[\gamma_7, k] = 4$. $[C_2, k] = 0$.

The images in Σ_3' of the fundamental curves k, C_2, γ_7 are, respectively, R_2': γ_2' C_3', R_4': $\gamma_2'^2$, R_{14}': $\gamma_2'^7$ $C_3'^3$, the C_3' being the image of S_3. Since the conjugate in I_8 of k is the singular plane β of C_2, the image of β is also R_2', which corresponds to the $K_4 \equiv S_3 + \beta$ of the general case.

$[R_2', R_4'] = \gamma_2'$ C_4', where the C_4' is the image of the ruled surface R_4 of Γ-lines meeting k and C_2.

$[R_2', R_{14}'] = \gamma_2'^7$ $C_3'^3$ $4\delta_i'$ l' where the $4\delta_i'$ correspond to the image lines in I_8 of the 4 points $[k, \gamma_7]$, and l' corresponds to the curve Δ_3, image in I_8 of the single point $[C_7, l]$, the l being the simple directrix of S_3 and the polar in Γ of k.

$[R_4', R_{14}'] = \gamma_2'^{14}$ $6\delta_i'$ C_{22}', where the $6\delta_i'$ correspond to the image conics in I_8 of the 6 points $[C_2, \gamma_7]$, and C_{22}' corresponds to the self-conjugate curve on the R_{22} of Γ-lines meeting C_2 and γ_7.

9. Suppose there are a pencil (A, α) and a regulus R on H_2 of parasitic lines. If the intersection of α and H_2 is a proper conic γ_2, the resulting transformation is essentially different from that discussed in Section 8. The fundamental curves are the conic γ_2, a line k in α, a γ_4 of genus 0 on H_2, and a space C_3. The lines of R are trisecants of γ_4. $[\gamma_4, C_3] = 6$, $[k, \gamma_2] = 2$, $[k, \gamma_4] = 0$, $[k, C_3] = 2$, $[\gamma_4, \gamma_2] = 4$, $[\gamma_2, C_3] = 0$. The C_3 passes through A.

The conjugates of planes in the I_8 are Φ_8: C_3^3 γ_4^2 $\gamma_2 k^2$ $4a_i$ $3b_i$. The a_i are bisecants of C_3 which meet γ_2 and γ_4, and the b_i are bisecants of γ_4 which meet C_3 and k.

The surface of invariant points is K_5: C_3^2 γ_4 k $4a_i$ $3b_i$.

The jacobian is composed of the conjugates in I_8 of the fundamental curves k, γ_2, C_3, and γ_4 which are, respectively, J_3: γ_4 C_3 k $3b_i$, J_4: γ_4 C_3^2 $4a_i$, J_{11}: γ_4^3 C_3^4 γ_2^2 k^2 $4a_i^2$ $3b_i$, and J_{10}: γ_4^2 C_3^4 γ_2 k^4 $4a_i$ $3b_i^2$.

The images in Σ_3' of (A, α) and R are, respectively, a line δ_1' meeting γ_2' and a conic C_2' meeting γ_2' in 2 points. The fundamental curves k, γ_2, C_3 and γ_4 (together with their conjugates in the I_8) correspond, respectively, to R_2': γ_2' δ_1', R_4': $\gamma_2'^2$ $\delta_1'^2$ C_2', R_6': $\gamma_2'^3$ δ_1', and R_8': $\gamma_2'^4$ $C_2'^3$.

10. If the surface S_μ is an S_4: k^2l^2, the double directrices are both necessarily fundamental. A composite form of such a surface is two quadrics H_1: k l C_4, H_2: k l γ_4, where the C_4 and γ_4 are of genus 1, and each meets both k and l in 2 points. Since $[C_4, \gamma_4] = 4$, the fundamental curve on S_4 is, in addition to k and l, a γ_8 of genus 5 which meets k and l each in 4 points.

The image in Σ_3' of S_4 is a curve C_4' of type [2, 2] on the quadric H_2' which corresponds in the Lie-Noether representation to the linear Γ-congruence with directrices k, l. To the fundamental lines k, l (of which each point of one corresponds in the I_7 to the whole of the other) correspond, respectively, the two reguli R_1', R_2' of H_2'. To γ_8 corresponds R_{16}': $\gamma_2'^8\ C_4'^2$.

$[H_2', R_{16}'] = \gamma_2'^8\ C_4'^2\ 8\delta_i'$, where the 8 δ_i' are the images in the Lie-Noether representation of the Γ-pencils with vertices at the 8 points of intersection of γ^8 with k and l.

11. If the surface S_μ is an S_4: C_3^2, a pair of quadrics H_1: $C_3\ \gamma_3$ and H_2: $C_3\delta_3$ may be taken as a composite case. $[C_3, \gamma_3] = 4$, $[C_3, \delta_3] = 4$, hence $[\gamma_3, \delta_3] = 2$. It follows that the fundamental curve on S_4 is, in addition to C_3, a C_6 of genus 1 which meets C_3 in 8 points. The residual fundamental curve not on S_4 is a quadrisecant k of C_6.

The image in Σ_3' of S_4 is a C_4' of genus 0. The images of k, C_3 and C_6 are R_2': γ_2', R_6': $\gamma_2'^3\ C_4'^2$, and R_{12}': $\gamma_2'^6\ C_4'^2$, respectively.

12. Suppose in Section 7 the fundamental line k is on the quadric J_2, hence meets C_5 in 2 points, P and Q. Since P and Q are in α and β, the conics C_2 and γ_2 must pass through them. The lines of the Γ-regulus on J_2 are now parasitic, being trisecants of C_5 meeting k. The surfaces α, β, J_2 form a composite S_4: $k^3\ C_9$, where the $C_9 \equiv C_5 + C_2 + \gamma_2$ has triple points in P and Q and is of genus 4. A transformation I_7 results, in which k is a singular invariant line, the Δ_3 associated with the Γ-pencil on any point of it being composed of the 3 generators of S_4 through the point.

The image in Σ_3' of S_4 is a C_4' of type [3, 1] on the H_2': γ_2' which corresponds in the Lie-Noether representation to the linear Γ-congruence containing S_4. The images of k and C_9 are the regulus R' of trisecants to C_4' on H_2' and R_{18}: $\gamma_2'^9\ C_4'^3\ 2\delta_i'^3$.

13. Suppose we have a pencil (A, α) and a ruled surface S_3 of parasitic lines, and suppose α and S_3 intersect in a proper conic C_2 and a generator g_1 of S_3, which necessarily passes through A. The fundamental curves are the conic C_2, a line l in α, the double edge k of S_3, a curve C_5 of genus 1 on S_3, and a line m on neither S_3 nor α. $[C_5, C_2] = 3$, $[C_5, k] = 3$, $[C_5, l] = 1$, $[C_5, m] = 3$, $[C_2, k] = 1$, $[C_2, l] = 2$, $[C_2, m] = 0$, $[k, l] = 0$, $[k, m] = 0$, $[l, m] = 1$. The point $[C_5, l]$ is on g_1 and C_5 passes through A.

The conjugates of planes in the resulting I_7 are Φ_7: $C_5^2\ C_2\ k\ l^2\ m^3\ a\ 3b_i\ 2c_i$. The line a meets C_5,k, l and m; the b_i are bisecants of C_5 meeting C_2 and m; and the c_i are trisecants of C_5 meeting l.

The surface of invariant points is K_4: $C_5\ l\ m^2\ a\ 3b_i\ 2c_i$.

The jacobian is composed of the conjugates in I_7 of the fundamental curves C_5, C_2, k, l and m which are, respectively, J_{13}: $C_5^4\ C_2^2\ k\ l^4\ m^5\ a\ 3b_i^2\ 2c_i^3$, J_8: $C_5\ m^2\ 3b_i$, J_1: $l\ m\ a$, J_3: $C_5\ k\ l\ m\ a\ 2c_i$, and J_4: $C_5\ C_2\ k\ l\ m^2\ a\ 3b_i$.

14. Suppose there exist two reguli R_1 on H_1 and R_2 on H_2 of parasitic lines. We shall assume that no part of the $[H_1, H_2] = C_4$ is a line of Γ. The fundamental curves are the C_4, lines k_1, k_2 on H_1 and k_3, k_4 on H_2, and the two transversals l_1, l_2 of k_1, k_2, k_3, k_4. None of the k_i belong to either R_1 or R_2. $[C_4, k_i] = 2$, $[k_i, k_j] = 0$, $[C_4, l_i] = 0$, $[k_i, l_i] = 1$.

The conjugates of planes in the I_7 are Φ_7: $C_4(k_1 \cdot\cdot k_4)^2(l_1 l_2)^3(a_1 \cdot\cdot a_4)$. The a_i are bisecants of C_4 meeting l_1 and l_2.

The surface of invariant points is K_4: $k_1 \cdot\cdot k_4\ l_1^2\ l_2^2\ a_1 \cdot\cdot a_4$.

The jacobian is composed of the conjugates in I_7 of the fundamental curves C_4, k_1, k_2, k_3, k_4, l_1 and l_2 which are, respectively J_8: $C_4(k_1 \cdot\cdot k_4)^2\ (l_1 l_2)^4(a_1 \cdot\cdot a_4)^2$, J_2: $k_1\ k_3\ k_4\ l_1\ l_2$, J_2: $k_2\ k_3\ k_4\ l_1\ l_2$, J_2: $k_1\ k_2\ k_3\ l_1\ l_2$, J_2: $k_1\ k_2\ k_4\ l_1\ l_2$, J_4: $C_4\ k_1 \cdot\cdot k_4\ l_1^2\ l_2\ a_1 \cdot\cdot a_4$, and J_4: $C_4\ k_1 \cdot\cdot k_4\ l_1\ l_2^2\ a_1 \cdot\cdot a_4$.

15. Given a C_5 of genus 1 and its surface of trisecants R_5: C_5^2. The R_5 belongs to a unique linear complex Γ'. There are ∞^4 cubic surfaces containing C_5. One surface F_3 of this system contains an arbitrary line l_1 of space. F_3: $C_5\ l_1\ l_2\ t_1 \cdot\cdot t_5$, where l_2 is the polar of l_1 in Γ' and $t_1 \cdot\cdot t_5$ are the trisecants of C_5 meeting l_1 and l_2. If l is in Γ' it is self-polar and F_3: $l\ t_1 \cdot\cdot t_5\ P^2$, where P is a point of l.* Conversely, for any point P of space there is a unique surface F_3: $C_5\ l\ t_1 \cdot\cdot t_5\ P^2$, where l is the only line of Γ' on F_3 through P. The other five lines of F_3 through P are bisecants of C_5. Associating the line l with the point P a (1, 1) correspondence is established between the lines of Γ' and the points of space, in which each line of Γ' is associated with a point on it. This correspondence will be denoted by M.

In M, a linear Γ-congruence with directrices l_1, $l_2 \sim F_3$: $C_5\ l_1\ l_2$. A Γ'-regulus $\sim C_4$, the residual base of a pencil of surfaces F_3: C_5. The C_4 is of genus 0 and meets C_5 in 10 points. A line of $R_5 \sim$ each of its points. Each line of the Γ'-pencil with vertex A on $C_5 \sim A$. An arbitrary Γ'-pencil $(A, \alpha) \sim$ a conic C_2 in α through A and through the five points $[\alpha, C_5]$.

In M^{-1}, a surface F_k: $C_5^h \sim$ a Γ'-congruence of order $(2k - 5h)$. A curve C_k having h points on $C_5 \sim$ a ruled surface R_{3k-h}: C_k. In particular, a

* See Colpitts, "On Twisted Quintic Curves," *American Journal of Mathematics*, Vol. 29 (1907), pp. 337-342; also Montesano, "Su la curva gobba di 5° ordine e di genere I," *Napoli Rendiconti*, Ser. 2, Vol. 2 (1888), pp. 181-188.

line l_1 not meeting $C_5 \sim R_3 : l_1\, l_2^2$, where l_2 is the polar of l_1 in Γ'. If l belongs to Γ', its image $R_3 : l^2$ is a Cayley scroll.

A surface F_3 of the system $\infty^4 \mid F_3 \mid : C_5$ determines a pair of lines l_1, l_2, polar in Γ'. A pencil of F_3 within the system determines a regulus of lines l_1, l_2, the associated regulus of which belongs to Γ'. A net of surfaces F_3 of the system determines a linear congruence of lines l_1, l_2, whose directrices are the Γ'-lines p_1, p_2 associated in M with the basis points P_1, P_2 of the net. Finally, a web of F_3 within the system determines a linear complex Γ of lines l_1, l_2. The complexes Γ' and Γ are in involution. If the web has a basis point, Γ is special, its directrix being the image in M^{-1} of the basis point.

Choose a web of $F_3 : C_5$ which does not have a basis point. A non-special linear complex Γ is then defined. A net of this web has two basis points P_1, P_2, either of which uniquely determines the other. For any point P_1 of space there exists such a net, hence a point P_2. Thus an I is established in space.

In M^{-1} a point P_1 of space corresponds to a line p_1 of Γ'. By the polarity in Γ, this line p_1 goes into a line p_2 which corresponds in M to P_2, conjugate to P_1 in the I. Since p_1, p_2 are polar in Γ, P_1P_2 is a line of Γ and I belongs to Γ.

A point P on C_5 corresponds in M^{-1} to the Γ'-pencil (P, π) which is transformed by the polarity in Γ into a Γ'-pencil (P', π'), where π' is the polar of P in Γ. Pencil (P', π') corresponds in M to a conic C_2 in π' through P. Hence C_5 is double on the surfaces Φ, conjugates of planes in I.

Each F_3 of the web is self-conjugate in I. Each curve C_4, base of a pencil in the web, is also self-conjugate in I. Since such a C_4 meets C_5 in 10 points, the transformation is of order 6.

The conjugate in I_6 of a generator t of R_5 is a point not in general on R_5. The locus of such points is a curve γ_5 of genus 1, simple on the surfaces Φ_6. $[C_5, \gamma_5] = 10$.

The conjugates of planes in I_6 are $\Phi_6 : C_5^2\, \gamma_5\, a_1 \cdot\cdot a_5$. The $a_1 \cdot\cdot a_5$ are trisecants of C_5 meeting γ_5. They are the only generators of R_5 which belong to Γ.

Let t_1 be a line of R_5. It, together with its conjugate T in I_6, belongs to a net of the web of F_3 defining Γ. A plane through t_1 meets C_5 in two points U and V, not on t_1. The line $TU = u$ has three fixed points on each F_3 of the net, hence there is within the net a pencil of F_3 which contains u. $\infty^1 \mid F_3 \mid : C_5\, u\, t_1\, t_2\, t_3$, where t_2 and t_3 are the trisecants of C_5 which meet u, other than t_1 and the two through U. The base of this pencil is a composite $C_4 \equiv u\, t_1\, t_2\, t_3$. Since t_1, t_2 and t_3 are fundamental, the line u must be self-conjugate. Since u contains the images of t_1, t_2, t_3, it is a trisecant of γ_5.

Hence, the trisecants of γ_5 all meet C_5 and are parasitic in I_6. They form a surface S_5: $C_5\ \gamma_5^2$.

The point-wise invariant surface is K_3: $C_5\ a_1 \cdot \cdot\ a_5$, which belongs to the original system $\infty^4 \mid F_3 \mid : C_5$, but not to the web defining Γ.

The images in I_6 of the fundamental curves C_5 and γ_5 are, respectively, J_{15}: $C_5^5\ \gamma_5^3\ (a_1 \cdot \cdot\ a_5)^3$ and R_5: $C_5^2\ a_1 \cdot \cdot\ a_5$. These surfaces form the jacobian of the transformation.

16. If the surface S_μ is S_5: C_6^2, the fundamental curve on S_5 is, in addition to C_6, a space C_3. The remainder of the composite C_{10} is a line k, not on S_5. The self-conjugate surfaces in the ∞^2 quadratic Γ-congruences which contain S_5 are $\infty^2 \mid R_3 \mid : k^2\ C_3$.

Conversely, suppose we are given a net of surfaces $\mid R_3 \mid : k^2\ C_3$, where k is a bisecant of C_3, and a non-special linear complex Γ. Through a point P_1 of space passes a pencil of R_3 within the net. The base of this pencil, aside from the base of the net, is a pair of generators p_1, p_2 with p_1 through P_1. The plane polar to P_1 in Γ cuts p_2 in a point P_2. The I thus defined belongs to Γ. The surfaces of the net $\mid R_3 \mid : k_2\ C_3$ are self-conjugate in I, and each line of a pair p_1, p_2 is the conjugate of the other.

The simple directrices k' of the net $\mid R_3 \mid : k^2\ C_3$ form a congruence $Q_{2,\,3}$ of order 2 and class 3. Each line of Γ contains a single pair of points conjugate in I, except the lines of $Q_{2,\,3}$ which belong to Γ, each of which contains ∞^1 pairs of conjugate points. There exists a ruled surface S_5: $V^3\ C_3$ of such lines, where V, on k, is the vertex of the single cubic cone belonging to the net $\mid R_3 \mid : k^2\ C_3$. The surface S_5 is of genius 0, hence contains a double curve C_6: V^3. The transformation I is of order 6, and in it the conjugates of planes are Φ_6: $k^3\ C_3^2\ C_6\ a_1\ a_2\ a_3$ the a_1, a_2, a_3 being the bisecants of C_6 which meet k and C_3.

If we take for k the line $x_1 = 0$, $x_2 = 0$, and for C_3 the curve $x_1 = \lambda^2\mu$, $x_2 = \lambda\mu^2$, $x_3 = \lambda^3$, $x_4 = \mu^3$, the surfaces R_3: $k^2\ C_3$ are $(x_1^2 - x_2x_3)(a_1x_1 + a_2x_2) + (x_2^2 - x_1x_4)(a_3x_1 + a_4x_2) = 0$. Select a net of these by imposing the relation $\sum_1^4 \rho_i\ a_i = 0$. Through a point $P_1(y)$ of space passes a pencil of this net. Two arbitrary surfaces of the pencil have in common, besides the k and C_3, two lines p_1, p_2, of which p_1 passes through P_1. The polar plane of P_1 in $\Gamma \equiv p_{12} - p_{34} = 0$ cuts p_2 in a point P_2, whose coordinates give the equations of the transformation. These are readily obtained, and are of order 6, as they should be.

17. A ruled surface S_6: C_3^2 is contained in a net of quadratic congruences, each pencil of which has for residual base a regulus R which has

four lines in common with the S_6. There is a net of such reguli. An arbitrary ray of Γ lies on a single regulus of the net; but each generator of S_6 belongs to ∞^1 such reguli.

Each linear Γ-congruence of the pencil determined by a regulus R of the net has in common with S_6, besides the four lines $[R, S_6]$, a pair of generators. An involution J is thus set up on S_6, which is independent of the R. If h, k are a pair of conjugate lines in J, each linear congruence Q_1: h, k contains a regulus R of the net, the generators of which establish a projectivity between the directrices of the congruence Q_1. These pairs of projective point ranges, which form a linear congruence with directrices h, k determine an I_5 belonging to Γ. The generators of S_6 are parasitic. The conjugates of planes in I_5 are Φ_5: $h^2\, k^2\, C_8\, a_1 \cdots a_4$. The $a_1 \cdots a_4$ are bisecants of C_8 which meet h and k.

18. If a ruled surface S_6: $C_9{}^2$ belongs to a non-special linear complex Γ, there are two points, A and B, triple on S_6 and on C_9. The Γ-pencils (A, α) and (B, β), together with the surface S_6, form the base of a pencil of quadratic congruences of Γ. In any congruence Q_2 of this pencil is a pair of pencils whose planes π_1, π_2, intersect in the line k joining A and B. These planes π_1, π_2 are met by the lines of Q_2 in pairs of points. Thus a quadratic correspondence is established between the planes in which the triads of points in which the planes meet C_9, other than in A and B, are fundamental. As Q_2 describes the pencil of quadratic congruences, the planes π_1, π_2 describe the pencil of planes on k. The resulting ∞^1 quadratic correspondences determine in space an I which belongs to Γ. The generators of S_6 are parasitic lines, hence the transformation is of order 5. The conjugates of planes are Φ_5: $k^3\, C_9\, a_1\, a_2$, where the a_1, a_2 are trisecants of C_9 which meet k.

19. Suppose that in the pencil of quadratic congruences of Section 18 there exists one, Q_2, formed by the lines of Γ which meet a conic C_2. The pair of planes π_1, π_2 which arises from Q_2 is composed of the plane ω of C_2 counted twice, the Γ-pencil (O, ω) being double in Q_2. Since the lines of (O, ω) are now parasitic, the transformation reduces to an I_4. The curve C_9 breaks up into the C_2: $A\, B$ and a C_7: $A^2\, B^2\, O$. The conjugates of planes are Φ_4: $k^2\, C_7\, a$, the line a being a trisecant of C_7 which meets k.

20. If there exists a ruled surface S_8 of parasitic lines, the transformation I_3 can have no invariant surface. Since the congruence of Γ-lines determined by the invariant points in the I is of order 4, the singular invariant curve, triple on S_8, must be a C_4. The generators of S_8 are bisecants of C_4, which is therefore of genus 1.

Suppose we have given a C_4 of genus 1, and a non-special linear complex Γ. The C_4 is the base of a pencil $|\, H_2\, |$ of quadric surfaces. The polar planes

of an arbitrary point P_1 with respect to the quadrics of the pencil meet in a line l. The polar plane of P_1 in Γ meets l in a point P_2. The pairs of points P_1, P_2 define an I which belongs to Γ. The C_4 is point-wise invariant in I. The congruence $Q_{2,6}$ of bisecants of C_4 contains a ruled surface S_8: C_4^3 of Γ-lines, each generator of which is parasitic. Hence the I is of order 3. The fundamental curve is, in addition to C_4, a C_6 of genus 3, double on S_8 and simple on the Φ_3, conjugates of planes in I_3. The jacobian of I_3 is J_8: C_6^3, conjugate to C_6.

No I of order < 3 can belong to a non-special linear complex.

II. *Transformations I Belonging to a Special Linear Complex.*

21. Suppose the pairs of conjugate points P_1, P_2 in an I lie on the rays of a special linear complex Γ, a single pair on an arbitrary ray. Let p denote the directrix of Γ.

The surface F_n, self-conjugate in a linear Γ-congruence with directrices p, q, contains simply the line q and meets an arbitrary line of the congruence in the pair of conjugate points lying on it. Hence if the surface is of order n, it contains the directrix p of Γ to multiplicity $(n-3)$. The n may be any integer greater than 3.

The self-conjugate surface in a bundle A of Γ is an F_{n-1}: $p^{n-4}\ A^{n-3}$. As the point A describes the line p, the surface F_{n-1} describes a pencil $|\ F_{n-1}\ |$, which determines the transformation. Through a point P_1 of space passes one surface F_{n-1}: $p^{n-4}\ A^{n-3}$ of the pencil. The line from P_1 through A meets F_{n-1} in one other point P_2. The pairs of points P_1, P_2 define the I belonging to Γ.

The self-conjugate curve in a Γ-pencil (A, α) with vertex A on p and plane α which does not contain p is a Δ_{n-1}: A^{n-3} of genus $(n-3)$. The self-conjugate curve in a Γ-pencil (B, β) with β: p is a Δ_3: B, of genus 1. The self-conjugate curve on a Γ-regulus is a C_{n+2} of genus $(n-1)$ which meets p in n points. Since a regulus is the base of a pencil of linear congruences, the fundamental curve, in addition to p, is a C_{5n-11} which meets p in $(5n-18)$ points. The curves p^{n-3} and C_{5n-11} form the base of a 4-fold system of surfaces F_n, hence the genus of C_{5n-11} is $(12n-38)$.

Each plane β through p is self-conjugate in I, the involution in it being a Geiser transformation having for fundamental points the 7 points $[\beta, C_{5n-11}]$ not on p. The I may be defined in this way.

A plane α through a line q skew to p and C_{5n-11} contains the Δ_{n-1} associated with the Γ-pencil in α. The curves q and Δ_{n-1} form the complete section of a surface F_n by α, hence any plane not through p meets C_{5n-11} in $(5n-11)$ points on a curve Δ_{n-1} which has an $(n-3)$-fold point on p.

The surface F_{n-1}: $p^{n-4}\ A^{n-3}$ associated with the Γ-bundle with vertex A on p meets the first polar of A with respect to F_{n-1}, besides in p, in a curve C_{5n-14}, locus of the double points in I which are on F_{n-1}. The locus of coincidences in a Geiser involution is a sextic curve passing twice through each of the seven fundamental points. It follows that the point-wise invariant surface of the I is a K_{3n-4}: $p^{3n-10}\ C^2{}_{5n-11}$.

The conjugate in I of a plane α which does not contain p is a surface Φ which is cut by α in a point-wise invariant C_{3n-4} and a self-conjugate Δ_{n-1}. Hence the transformation is of order $(4n-5)$. The conjugates of planes in the I_{4n-5} are Φ_{4n-5}: $p^{4n-13}\ C_{5n-11}{}^3\ (15n-40)a_i$. The parasitic lines a_i are trisecants of C_{5n-11} which meet p.

The conjugates in I_{4n-5} of the fundamental curves p and C_{5n-11} are, respectively, J_{4n-6}: $p^{4n-14}\ C_{5n-11}{}^3\ (15n-40)a_i$, and J_{12n-18}: $p^{12n-39}\ C_{5n-11}{}^8\ (15n-40)a_i{}^3$. These surfaces form the jacobian of the transformation.

22. As in the case of transformations I belonging to a non-special linear complex, the fundamental curve C_{5n-11} may become composite in such a way as to admit of ∞^1 trisecants meeting p which form a ruled surface S_μ. The I is then of order $(4n-5-\mu)$, the surface S_μ being a part of each of the surfaces Φ, conjugates of planes in the I. The surface of invariant points is also reduced in order to $(3n-4-\mu)$.

A simple directrix curve of S_μ may or may not be fundamental. A double directrix curve of S_μ is necessarily fundamental. No directrix curve other than p of multiplicity > 3 can exist on S_μ.

A few of the possibilities will be discussed.

23. Suppose the surface S_μ of parasitic lines is a Γ-pencil (A, α) with vertex A on p and plane α not containing p. The surfaces F_n associated in the transformation with the ∞^2 linear Γ-congruences which contain the pencil (A, α) break up into the plane α and a net of surfaces $|F_{n-1}|$, of which the base is the line p^{n-3} and that part of the fundamental C_{5n-11} not in the plane α. A pencil of the net associated with the linear Γ-congruences whose directrices form a pencil (M, α) has for residual base a conic Δ_2, which, together with the parasitic line MA, forms the self-conjugate Δ_3 in the plane (M, p). Hence the base of the net $|F_{n-1}|$ is, in addition to p, a C_{4n-10} of genus $(6n-20)$. The fundamental curve in α is then γ_{n-1}: A^{n-4} of genus $(2n-7)$, which meets C_{4n-10} in $(4n-10)$ points. The C_{4n-10} meets p in $(4n-14)$ points.

The I_{4n-6} is completely determined by the net of surfaces $\infty^2\ |F_{n-1}|$: $p^{n-3}\ C_{4n-10}$, and the complex Γ.

The conjugates of planes in the I are Φ_{4n-6}: $p^{4n-13}\ C_{4n-10}{}^3\ \gamma_{n-1}{}^2\ (12n-33)a_i$. The a_i are bisecants of C_{4n-10} which meet γ_{n-1} and p.

The surface of invariant points is K_{3n-5}: $p^{3n-10}\ C_{4n-10}{}^2\ \gamma_{n-1}\ (12n-33)a_i$.

The jacobian of the transformation is composed of the conjugates in I of the fundamental curves p, C_{4n-10}, and γ_{n-1}, which are, respectively,

$$J_{4n-7}\colon\ p^{4n-14}\ C_{4n-10}{}^3\ \gamma_{n-1}{}^2\ (12n-33)\ a_i,$$
$$J_{8n-14}\colon\ p^{8n-26}\ C_{4n-10}{}^5\ \gamma_{n-1}{}^4\ (12n-33)\ a_i{}^2,\ \text{and}$$
$$J_{4n-7}\colon\ p^{4n-13}\ C_{4n-10}{}^3\ \gamma_{n-1}\ (12n-33)a_i.$$

24. The surface of parasitic lines may also be a pencil (B, α) with vertex B not on p and plane β through p. The base of the net of surfaces $|F_{n-1}|$ associated with the ∞^2 linear Γ-congruences which contain the pencil (B, β) is the line p^{n-4} and that part of the fundamental C_{5n-11} not in β. The residual base of a pencil of the net is a Δ_{n-2}. The fundamental curves are p, a space C_{5n-13} of genus $(12n-43)$ which meets p in $(5n-20)$ points, and a conic γ_2 in β which meets C_{5n-13} in six points. The seventh point $[C_{5n-13}, \beta]$ not on p is the vertex β of the pencil of parasitic lines.

The I_{4-n6} is determined by the net of surfaces F_{n-1} and the complex Γ.

The conjugates of planes in I are Φ_{4n-6}: $p^{4n-14}\ C_{5n-13}{}^3\ \gamma_2{}^2\ (15n-45)a_i$. The a_i are trisecants of C_{5n-13} which meet p.

The surface of invariant points is K_{3n-5}: $p^{3n-11}\ C_{5n-13}{}^2\ \gamma_2\ (15n-45)\ a_i$.

The jacobian of the transformation is composed of the conjugates of the fundamental curves p and C_{5n-13}, which are, respectively,

$$J_{4n-7}\colon\ p^{4n-15}\ C_{5n-13}{}^3\ \gamma_2{}^2\ (15n-45)\ a_i \text{ and}$$
$$J_{12n-21}\colon\ p^{12n-42}\ C_{5n-13}{}^8\ \gamma_2{}^6\ (15n-45)\ a_i{}^3.$$

The fundamental conic γ_2 is parasitic in the I, each point of it corresponding to the whole conic.

25. Suppose the composite C_{5n-11} admits of a regulus R of trisecants which meet p. Such a regulus, if composite, is made up of two pencils (A, α) and (B, β) with a common line l, one being one type, the other of the other, of those just discussed. Since the fundamental γ_{n-1} and γ_2 of the two pencils meet in the pair of conjugate points on l, the fundamental curve on the quadric S_2 containing R is, in addition to the line p, a γ_{n+1} of genus $(2n-6)$ which meets p in $(n-2)$ points. The residual fundamental curve, not on S_2, is a C_{4n-12} of genus $(6n-23)$ which meets p in $(4n-16)$ points and γ_{n+1} in $(4n-8)$ points.

The I_{4n-7} is determined by the pencil of surfaces $|F_{n-2}|$: $p^{n-4}\ C_{4n-12}$ which is associated with the pencil of linear Γ-congruences containing R.

The conjugates of planes in I are Φ_{4n-7}: $p^{4n-14}\ C_{4n-12}{}^3\ \gamma_{n+1}{}^2\ (12n-36)a_i$. The a_i are bisecants of C_{4n-12} which meet γ_{n+1} and p.

The surface of invariant points is K_{3n-6}: $p^{3n-11}\ C_{4n-12}{}^2\ \gamma_{n+1}\ (12n-36)\ a_i$.

The jacobian is composed of the conjugates of the fundamental curves p, C_{4n-12} and γ_{n+1}, which are respectively,

$$J_{4n-8}:\ p^{4n-15}\ C_{4n-12}{}^3\ \gamma_{n+1}{}^2\ (12n-36)\ a_i,$$
$$J_{8n-16}:\ p^{8n-28}\ C_{4n-12}{}^5\ \gamma_{n+1}{}^4\ (12n-36)\ a_i{}^2,$$
$$J_{4n-8}:\ p^{4n-14}\ C_{4n-12}{}^3\ \gamma_{n+1}\ (12n-36)\ a_i.$$

26. Suppose the surface of parasitic lines is S_μ: $p^{\mu-1}\ q$, $\mu \geqq 1$, where q is skew to p. It is easy to show by mathematical induction that the fundamental curves, aside from p are $\gamma_{n+2\mu-3}$ of genus $(2n+\mu-8)$ on S_μ which meets p in $(n+2\mu-6)$ points, and $C_{4n-2\mu-8}$ of genus $(6n-3\mu-17)$, not on S_μ, which meets p in $(4n-2\mu-12)$ points and $\gamma_{n+2\mu-3}$ in $(4n+2\mu-12)$ points.

A plane π through p meets $C_{4n-2\mu-8}$ in four points and $\gamma_{n+2\mu-3}$ in three points, aside from intersections on p. These seven points are fundamental points for a Geiser involution in π. As π describes the pencil p, an $I_{4n-\mu-5}$ is generated in which the conjugates of planes are $\Phi_{4n-\mu-5}$: $p^{4n-\mu-12}\ C_{4n-2\mu-8}{}^3\ \gamma_{n+2\mu-3}{}^2\ (12n-3\mu-30)\ a_i$. The a_i are bisecants of $C_{4n-2\mu-8}$ which meet $\gamma_{n+2\mu-3}$ and p.

The surface of invariant points is $K_{3n-\mu-4}$: $p^{3n-\mu-9}\ C_{4n-2\mu-8}{}^2\ \gamma_{n+2\mu-3}\ (12n-3\mu-30)\ a_i$.

The jacobian of the transformation is composed of the conjugates of the fundamental curves p, $C_{4n-2\mu-8}$ and $\gamma_{n+2\mu-3}$ which are, respectively:

$$J_{4n-\mu-6}:\ p^{4n-\mu-13}\ C_{4n-2\mu-8}{}^3\ \gamma_{n+2\mu-3}{}^2\ (12n-3\mu-30)\ a_i,$$
$$J_{8n-2\mu-12}:\ p^{8n-2\mu-24}\ C_{4n-2\mu-8}{}^5\ \gamma_{n+2\mu-3}{}^4\ (12n-3\mu-30)\ a_i{}^2,$$
$$J_{4n-\mu-6}:\ p^{4n-\mu-12}\ C_{4n-2\mu-8}{}^3\ \gamma_{n+2\mu-3}\ (12n-3\mu-30)\ a_i.$$

27. If two Γ-pencils (B, β) and (B', β') with vertices not on p are parasitic, the ∞^3 conics which contain B and B' and meet p are the directrices of ∞^3 Γ-congruences $Q_{1,2}$ containing (B, β) and (B', β'). The associated web of self-conjugate surfaces is, aside from β and β', a web $|F_{n-1}|$: $p^{n-4}\ C_{5n-15}$. If, in particular, the fundamental conics γ_2 in β and γ_2' in β' intersect in a pair of points O_1 and O_2, the C_{5n-15} meets p in $(5n-20)$ points, is of genus $(12n-46)$, passes through B and B', and meets γ_2 and γ_2' each in four points. The points O_1 and O_2 are $(n-3)$fold basis points in the web $|F_{n-1}|$. Now set $n=4$ and we have a well-known I_9.*

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* See Sharpe and Snyder, "Certain Types of Involutorial Space Transformations," *Transactions of the American Mathematical Society*, Vol. 21 (1920), p. 58. Also C. Moffa, "Su alcune corrispondenze birazionali involutorie dello spazio dotate di un sistema lineare di dimensione tre di superficie del terzo ordine unite," *Napoli Dissertation* (1923), pp. 11-14.

On the Inverse Problem in the Calculus of Variations.

THOMAS H. RAWLES.

In a previous paper we have applied the theory connected with the invariant integral to the inverse problem in the calculus of variations.* The plan of attack was to determine in the first place the general form of integrand function which results from a prescribed relation between the slope of the extremal and the normal to the transversal curves or surfaces. This determination involves an arbitrary function of the coördinates which in turn may be determined by assigning a family of curves as the extremals of the problem. Conditions were given under which an arbitrary family of curves may be taken as the extremals of a problem involving a given transversality.

The object of the present paper is to determine directly the most general form of the integrand function which may have as extremals a given two parameter family of curves, a problem which originates with Darboux.† By approaching this problem from the standpoint of the invariant integral we can obtain the solution by somewhat simpler processes than those involved in the method of Darboux.

We have first to show that if $y(x, a, b)$ is a general solution of the Euler equation arising from a problem in two dimensions, a and b being any two constants whatever, we can construct a function having as arguments x, y, and one of these constants, $W(x, y, a)$ say, such that $(\partial/\partial a)W(x, y, a)=b'$ is a general solution of the Euler equation.

Such a function may be obtained from the definite integral

$$(1) \qquad J(x, a, b) = \int_X^x f\,[x, y(x, a, b), y_x(x, a, b)\,]\,dx,$$

where $f(x, y, y')$ is the integrand of the integral minimized by the curves $y = y(x, a, b)$. The lower limit, X, is itself a function of a and b such that the equations $x = X(a, b)$, $y = Y\,[X(a, b), a, b]$ represent a transversal curve associated with $y = y(x, a, b)$, a being assigned a fixed value, and b being taken as the parameter.

Now if $y_b(x, a, b) \neq 0$ for a certain set of values of the arguments we

* Rawles, *Transactions of the American Mathematical Society*, Vol. 30, pp. 765-784.

† Darboux, *Théorie des Surfaces*, Vol. III, paragraphs 604, 605, 606.

may solve the equation $y = y(x, a, b)$ for b and obtain $b = B(x, y, a)$. When this result is substituted in the left member of (1) the result is

$$J[x, a, B(x, y, a)] = W(x, y, a). \tag{2}$$

We calculate the derivatives of the function W by differentiating (1) and obtain the familiar expressions

$$\partial W/\partial x = f(x, y, p) - pf_{y'}(x, y, p),\ \partial W/\partial y = f_{y'}(x, y, p). \tag{3}$$

where $p(x, y, a) = y_x[x, a, B(x, y, a)]$.

In the same manner the derivatives of W_a are given by

$$\partial W_a/\partial x = -pf_{y'y'}(x, y, p)p_a,\ \partial W_a/\partial y = f_{y'y'}(x, y, p)p_a. \tag{4}$$

We now consider the curve defined by

$$W_a = b'. \tag{5}$$

Along this curve we must have $f_{y'y'}p_a(dy - dx\,p) = 0$. If $f_{y'y'} \equiv 0$ the integrand function contains the derivative y' only as a linear term. This would indicate a degenerate form of the problem which we do not consider. Also $p_a \not\equiv 0$; for if it were $y(x, a, b)$ would not be a general solution of a second order differential equation.*

Limiting ourselves, then, to those values of x, y, and y' for which $f_{y'y'} \neq 0$, $p_a \neq 0$, it follows that along the curve defined by (5) $dy/dx = p(x, y, a)$.

By the differentiation of (2) we can actually calculate W_a. We find

$$W_a = -f_{y'}Y_a - [f - y_x(X, a, b)f_{y'}]X_a,$$

where the arguments of f and $f_{y'}$ are X, Y, and $y_x(X, a, b)$ and b is replaced by $B(x, y, a)$. When the function W_a reduces simply to $B(x, y, a)$ the constants a and b are said to be canonical. In general, however, we obtain $W = \phi\,[a, B(x, y, a)]$; and it can be shown further that, under the assumptions made, $\partial\phi/\partial b \neq 0$.

The theory which we have outlined leads us to a method of obtaining the integrand functions which are minimized by a given two parameter family of curves. Let us take as our extremals the family represented by $y = y(x, a, b)$ and solve for b finding $b = B(x, y, a)$. We now form an arbitrary function, $\phi\,[a, B(x, y, a)]$, which may be regarded as the derivative of the transversal function, $W(x, y, a)$.

When we integrate with respect to a we obtain

$$W(x, y, a) = \int \phi[a, B(x, y, a)]\,da + \theta(x, y), \tag{6}$$

* Bolza, *Vorlesungen über Variationsrechnung*, p. 138.

where θ is an arbitrary function. Also we may differentiate $B(x, y, a) = b$ with respect to x considering y as a function of x, and then solve for a, finding $a = A(x, y, p)$.

On the other hand we see by (3) that

$$f(x, y, p) = \partial W/\partial x + p\,\partial W/\partial y, \tag{7}$$

where a is replaced by $A(x, y, p)$ after the differentiation. This gives us finally,

$$\begin{aligned} f(x, y, p) = (\partial/\partial x) \int \phi[a, B(x, y, a)]da + p(\partial/\partial y) \int \phi[a, B(x, y, a)]da \\ + \theta_x(x, y) + p\theta_y(x, y), \\ a = A(x, y, p). \end{aligned} \tag{8}$$

Conversely, to show that (8) has as its extremals the given family of curves we have only to reverse the argument. If we form the Hamiltonian equation associated with (8) we have in (6) a solution already at hand.* It then follows that the equations of the extremals are given by

$$(\partial/\partial a)W(x, y, a) = \phi\,[a, B(x, y, a)] = b'.\dagger$$

Finally, to restore the original constants we put $b' = \phi(a, b)$ and the equations of the extremals assume the form

$$B(x, y, a) = b. \tag{9}$$

To illustrate the method we shall take the problem of determining the integrand function which has as its extremals straight lines. If the family of extremals is $y = ax + b$, $B(x, y, a) = y - ax$, and $a = p$. For the integrand function we find

$$\begin{aligned} f(x, y, p) = (\partial/\partial x) \int \phi(a, y - ax)da + p(\partial/\partial y) \int \phi(a, y - ax)da \\ + \theta_x(x, y) + p\theta_y(x, y), \\ a = p. \end{aligned} \tag{8a}$$

If we differentiate with respect to x and y under the integral signs (8a) becomes

$$\begin{aligned} f(x, y, p) = \int (p - a)\phi'(a, y - ax)da + \theta_x(x, y) + p\theta_y(x, y), \\ a = p, \end{aligned}$$

where ϕ' indicates the derivative of ϕ with respect to the second argument. This is the form of solution given by Darboux.‡

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* Bolza, *loc. cit.*, p. 132.

† Bolza, *loc. cit.*, p. 138.

‡ Darboux, *loc. cit.*, paragraph 606; also, Bolza, *loc. cit.*, p. 39.

An Application of the Laguerre Method for the Representation of Imaginary Points.

By B. M. Turner.

1. *Introduction.* In an earlier paper * the writer directed attention to the fact that while three collinear real points of inflexion impose but five conditions on a real † non-singular plane cubic curve, and hence leave the curve with four degrees of freedom; still not one of the six imaginary points of inflexion may be chosen arbitrarily. The statement of the fact was followed by a discussion of the positions of the imaginary points of inflexion and critic centers for the four-fold infinite system of cubics. This paper shows that the variable imaginary inflexions and critic centers, represented by real point-pairs in accordance with the Laguerre method for the representation of imaginary points, describe unique systems of curves; and brings out more clearly the relations of the sets of points.

2. *The Laguerre Method.*‡ In the Laguerre representation of points in a plane the line at infinity, $z = 0$, and the circular points

$$(1, i, 0), \quad (1, -i, 0), \qquad\qquad i = (-1)^{\frac{1}{2}},$$

are fixed. Two lines, extending into the finite part of the plane and passing through the circular points in the order given, are called the first and second minimal lines of their intersection. Each real point is represented by itself; and each finite imaginary point by the pair of real points which lie on its minimal lines, the first and second point of the pair lying respectively on the first and second minimal line. The same pair of points in the reverse order represents the conjugate imaginary point.

For a pair of imaginary points the intersections of the first minimal line of each one by the second minimal line of the other form the representative real point-pair. For the imaginary point $(a + bi,\ c + di,\ 1)$ the minimal lines are

$$\begin{vmatrix} x & y & z \\ a + bi & c + di & 1 \\ 1 & \pm i & 0 \end{vmatrix} = 0;$$

* *Annals of Mathematics*, Vol. 33, No. 4 (June, 1922).

† Real in the sense that all coefficients of the equation are real.

‡ *Oeuvres de Laguerre*, t. 2, Paris (1905), pp. 88-108. Also see Coolidge, *Geometry of the Complex Domain*, Oxford (1924), p. 85.

and the real points on these lines $(a-d,\ c+b,\ 1)$, $(a+d,\ c-b,\ 1)$ form the representative point-pair. The pair in the opposite order

$$(a+d,\ c-b,\ 1),\ (a-d,\ c+b,\ 1)$$

represent the conjugate point $(a-bi,\ c-di,\ 1)$.

By definition a chain * is a system of collinear points which satisfy the following conditions:

(1) The cross-ratios of any four are real.

(2) With three arbitrarily chosen points of the system there exists a fourth which makes any real cross-ratio other than zero, one and infinity.

In the Laguerre representation a chain appears as two real lines or circles.

3. *Chains of Inflexions.* By the earlier paper the system of real cubics with $I_1\ (0,1,-1)$, $I_2\ (-1,0,1)$, $I_3\ (1,-1,0)$ as points of inflexion and $\Sigma\ (1,1,1)$ as fixed critic center common to the three real harmonic polars, has an equation

$$(x+y+z)(x^2+y^2+z^2+yz+zx+xy) + \lambda(\alpha x+y+z)(x+\alpha y+z)(x+y+\alpha z)=0.$$

The real harmonic polars are

$$h_1 \equiv y-z=0,\quad h_2 \equiv z-x=0,\quad h_3 \equiv x-y=0;$$

and the imaginary inflexions are

$$\begin{array}{ll} (\omega^2-\omega,\ 1-\alpha\omega^2,\ \alpha\omega-1), & (\omega-\omega^2,\ 1-\alpha\omega,\ \alpha\omega^2-1); \\ (\alpha\omega-1,\ \omega^2-\omega,\ 1-\alpha\omega^2), & (\alpha\omega^2-1,\ \omega-\omega^2,\ 1-\alpha\omega); \\ (1-\alpha\omega^2,\ \alpha\omega-1,\ \omega^2-\omega), & (1-\alpha\omega,\ \alpha\omega^2-1,\ \omega-\omega^2); \end{array}$$

where $1, \omega, \omega^2$ are the cube roots of unity.

The cross-ratios of the four points

$$(\omega^2-\omega,\ 1-\alpha_j\omega^2,\ \alpha_j\omega-1),\qquad (j=1,2,3,4),$$

are those of the group containing

$$(\alpha_1-\alpha_2)(\alpha_3-\alpha_4)/(\alpha_1-\alpha_4)(\alpha_3-\alpha_2),$$

all real. With three points

$$(\omega^2-\omega,\ 1-\alpha_j\omega^2,\ \alpha_j\omega-1),\qquad (j=1,2,3),$$

* Von Staudt, *Beiträge zur Geometrie der Lage*, Part II, Nüremburg (1858), pp. 137 ff. Coolidge, *loc. cit.*, p. 36.

there exists a fourth which makes a real cross-ratio r; namely,

$$(\omega^2 - \omega, \quad 1 - \alpha_4\omega^2, \quad \alpha_4\omega - 1),$$

with $\alpha_4 = [r\alpha_1(\alpha_3 - \alpha_2) + \alpha_3(\alpha_2 - \alpha_1)]/[r(\alpha_3 - \alpha_2) + (\alpha_2 - \alpha_1)]$.

Then as α varies, the points $(\omega^2 - \omega, 1 - \alpha\omega^2, \alpha\omega - 1)$ form a chain on the line $x + \omega y + \omega^2 z = 0$, and similarly five other chains are described. The totality is three pairs of chains, the mates of each pair lying one on each of the lines

$$x + \omega y + \omega^2 z = 0, \text{ and } x + \omega^2 y + \omega z = 0,$$

which are equiharmonic to the three real harmonic polars. Hence *the imaginary points of inflexion for the doubly infinite system of real cubics, determined by three real points of inflexion and the critic center common to the three real harmonic polars, form three pairs of chains, the mates of each pair lying one on each of the Hessian lines of the real harmonic polars.*

With the fixed line at infinity, $z = 0$, I_1 and I_2 are finite points and I_3 the point at infinity on the line

$$I_1 I_2 \equiv x + y + z = 0.$$

By the Laguerre method for the representation of imaginary points in a plane, each pair of the imaginary points of inflexion is represented by a pair of real points. The pair

$$(\omega^2 - \omega, 1 - \alpha\omega^2, \alpha\omega - 1), \quad (\omega - \omega^2, 1 - \alpha\omega, \alpha\omega^2 - 1)$$

is represented by

$$\{\pm 3^{\frac{1}{2}}\alpha^2 + [3 \pm 2(3^{\frac{1}{2}})]\alpha, \quad -\alpha^2 + (2 \pm 3^{\frac{1}{2}})\alpha + 2 \pm 2(3^{\frac{1}{2}}), \quad -2(\alpha^2 + \alpha + 1)\},$$

the upper signs throughout giving one of the real points and the lower signs the other.* As α varies, these real points describe the two circles

$$x^2 + y^2 - xz \pm 3^{\frac{1}{2}}yz - (1 \pm 3^{\frac{1}{2}})z^2 = 0.$$

Similarly the pair of imaginary inflexions

$$(\alpha\omega - 1, \omega^2 - \omega, 1 - \alpha\omega^2), \quad (\alpha\omega^2 - 1, \omega - \omega^2, 1 - \alpha\omega)$$

is represented by the real points

* This convention is observed wherever double signs are used in the representation of pairs of points or circles.

$$\{-\alpha^2 + (2 \pm 3^{\frac{1}{2}})\alpha + 2 \pm 2(3^{\frac{1}{2}}), \quad \pm 3^{\frac{1}{2}}\alpha^2 + [3 \pm 2(3^{\frac{1}{2}})]\alpha, \quad -2(\alpha^2 + \alpha + 1)\},$$

which for varying α describe the circles

$$x^2 + y^2 \pm 3^{\frac{1}{2}}xz - yz - (1 \pm 3^{\frac{1}{2}})z^2 = 0.$$

The third pair of imaginary inflexions

$$(1 - \alpha\omega^2, \ \alpha\omega - 1, \ \omega^2 - \omega), \quad (1 - \alpha\omega, \ \alpha\omega^2 - 1, \ \omega - \omega^2)$$

is represented by the point-pair

$$[(1 \pm 3^{\frac{1}{2}})\alpha + 2, \ (1 \pm 3^{\frac{1}{2}})\alpha + 2, \ \mp 2(3^{\frac{1}{2}})],$$

whose locus as α varies is the line $x - y = 0$.

Thus it is found that *the Laguerre representation of the imaginary points of inflexion for the doubly infinite system of cubic curves with fixed real points of inflexion, two finite and one at infinity, and a fixed critic center, the one common to the three real harmonic polars, consists of two pairs of circles and a straight line counted twice.*

The straight line in the representation indicates a specialization due to having one fixed real inflexion at infinity. To obtain a more general result, the case when all three real inflexions are finite must be considered. It is here shown that the representatives of the three pairs of chains are not independent. They are related to each other and also to the fixed critic center, real harmonic polars and other associates of the cubic. These relations may be anticipated in the generalization.

4. *Chains of Critic Centers.* Before studying the more general form, the representatives of the critic centers for the system of cubics with one real inflexion at infinity may well be considered.

The fixed elements uniquely determine two imaginary critic centers, the Hessian points of I_1, I_2, I_3,

$$(\omega^2, 1, \omega), \quad (\omega, 1, \omega^2).$$

These are represented by two real points

$$H^{(1)}(1 + 3^{\frac{1}{2}}, 1 + 3^{\frac{1}{2}}, -2), \quad H^{(2)}(1 - 3^{\frac{1}{2}}, 1 - 3^{\frac{1}{2}}, -2)$$

on the line $x - y = 0$ where the four circles intersect in pairs.

The other three real critic centers are

$$V_1(\alpha + 1, -1, -1), \quad V_2(-1, \alpha + 1, -1), \quad V_3(-1, -1, \alpha + 1),$$

which as α varies describe the real harmonic polars h_1, h_2, h_3.

The remaining three pairs of imaginary critic centers are

$$(\alpha - 2\omega^2 + 1,\ \alpha\omega^2 - 1,\ \alpha\omega^2 - 1), \quad (\alpha - 2\omega + 1,\ \alpha\omega - 1,\ \alpha\omega - 1);$$
$$(\alpha\omega^2 - 1,\ \alpha - 2\omega^2 + 1,\ \alpha\omega^2 - 1), \quad (\alpha\omega - 1,\ \alpha - 2\omega + 1,\ \alpha\omega - 1);$$
$$(\alpha\omega^2 - 1,\ \alpha\omega^2 - 1,\ \alpha - 2\omega^2 + 1), \quad (\alpha\omega - 1,\ \alpha\omega - 1,\ \alpha - 2\omega + 1).$$

These are represented by the real point-pairs

$$[-\alpha^2 - 7\alpha - 4,\ \alpha^2(2 \mp 3^{\frac{1}{2}}) + \alpha(2 \mp 3^{\frac{1}{2}}) + 2 \pm 2(3^{\frac{1}{2}}),\ 2(\alpha^2 + \alpha + 1)],$$
$$[\alpha^2(2 \pm 3^{\frac{1}{2}}) + \alpha(2 \pm 3^{\frac{1}{2}}) + 2 \mp 2(3^{\frac{1}{2}}),\ -\alpha^2 - 7\alpha - 4,\ 2(\alpha^2 + \alpha + 1)],$$
$$[\alpha^2(1 \pm 3^{\frac{1}{2}}) + \alpha(7 \pm 3^{\frac{1}{2}}) + 4 \mp 2(3^{\frac{1}{2}}),$$
$$\alpha^2(1 \mp 3^{\frac{1}{2}}) + \alpha(7 \mp 3^{\frac{1}{2}}) + 4 \pm 2(3^{\frac{1}{2}}),\ -2(\alpha^2 + 4\alpha + 7)],$$

which, independent of α, form the following three pairs of circles:

$$x^2 + y^2 + xz - (2 \pm 3^{\frac{1}{2}})yz - (1 \mp 3^{\frac{1}{2}})z^2 = 0,$$
$$x^2 + y^2 - (2 \mp 3^{\frac{1}{2}})xz + yz - (1 \pm 3^{\frac{1}{2}})z^2 = 0,$$
$$2(x^2 + y^2) - (1 \mp 3^{\frac{1}{2}})xz - (1 \pm 3^{\frac{1}{2}})yz - 2z^2 = 0.$$

These circles all pass through the point $\Sigma(1, 1, 1)$, with the second intersections of the mates of the pairs where the real harmonic polars meet the line through the real inflexions, that is at the Jacobian points of the three real inflexions. Further relations of these circles to each other and to the representations of the imaginary inflexions appear and hence may be anticipated in the generalization.

5. *Chains of Hessian Points.* For the further study let the three real points of inflexion be

$$I_1: (0, 1, -1), \quad I_2: (-1, 0, 1), \quad I_3: (a, -1, 1 - a),$$

where a may have any real value except zero and infinity, and I_3 is finite except when $a = 1$, which gives the special case that has been considered.

The Hessian points for I_1, I_2, I_3 are

$$(\omega^2 a, 1, -1 - \omega^2 a), \quad (\omega a, 1, -1 - \omega a),$$

represented by the real points

$$H^{(i)}: [2a^2 - a(1 \mp 3^{\frac{1}{2}}), -a(1 \mp 3^{\frac{1}{2}}) + 2, -2(a^2 - a + 1),^* \quad (i = 1, 2).$$

As a varies the Hessian points form a pair of chains and the representatives describe a pair of circles

$$H_c^{(i)}: \pm 3^{\frac{1}{2}}(x^2 + y^2) + (1 \pm 3^{\frac{1}{2}})xz + (1 \pm 3^{\frac{1}{2}})yz + z^2 = 0,^* \quad (i = 1, 2).$$

* Here the upper sign is to be used when $i = 1$, the lower sign when $i = 2$.

These circles both pass through the points I_1 and I_2 and each passes through the center of the other.

Hence *the locus in a plane of the Laguerre representatives of the Hessian points with respect to two fixed real points and a collinear variable third real point is uniquely defined as two circles through the two fixed points such that each circle passes through the center of the other.*

As a takes all positive values from zero to infinity, the variable I_3 runs along the line segments exterior to I_1 and I_2; and the representatives of the corresponding Hessian points describe the outer arcs of the circles. As a takes all negative values from zero to infinity, I_3 runs along the interior segment from I_1 to I_2 and the representatives of the Hessians describe the inner arcs of the circles.

For I_3: $(a, -1, 1-a)$ the corresponding Hessian representatives are cut out on the circles by the line

$$x - y + (a^2 - 1)/(a^2 - a + 1) = 0.$$

The maximum and minimum values of $(a^2-1)/(a^2-a+1)$ are $\pm 2/3^{\frac{1}{2}}$; that is, $x - y \pm 2/3^{\frac{1}{2}} = 0$ are tangent lines to the circles, and each of the lines between the tangents cuts out two of the real point-pairs.

When $(a^2-1)/(a^2-a+1) = \pm 2/3^{\frac{1}{2}}$ the points I_3 are

$$M^{(1)}: (1 + 3^{\frac{1}{2}}, 1 - 3^{\frac{1}{2}}, -2), \quad M^{(2)}: (1 - 3^{\frac{1}{2}}, 1 + 3^{\frac{1}{2}}, -2).$$

As I_3 runs along the line segments exterior to these points, the representatives of the corresponding Hessian points describe the outer semicircles between the points of contact of the tangents. As I_3 runs along the line segment between these points the representatives of the Hessians describe the inner semicircles between the points of contact.

The Marie * representation of the imaginary points is obtained from the Laguerre by rotating each point-pair around the midpoint of the included line segment through an angle of ninety degrees. Thus imaginary points on a line are represented by pairs of real points on that line; and the Laguerre and Marie representatives of an imaginary point form the vertices of a square.

The points

$$M^{(1)}: (1 + 3^{\frac{1}{2}}, 1 - 3^{\frac{1}{2}}, -2), \quad M^{(2)}: (1 - 3^{\frac{1}{2}}, 1 + 3^{\frac{1}{2}}, -2)$$

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* Marie, *Réalisation et usage des formes imaginaires en géométrie*, Paris (1891); Mouchot, *Les branches de la géométrie supérieure*, Paris (1892); Study, *Ausgewählte Gegenstände der Geometrie*, Leipzig (1911); Coolidge, *Geometry of the Complex Domain*, Oxford (1924).

which determine the Hessian points whose Laguerre representatives are the points of contact of the tangent lines are the Marie representatives of the Hessian points when $a = 1$, that is, when I_3 is the point at infinity.

It follows directly that the *locus in three-space of the Laguerre representatives of the Hessian points with respect to two fixed real points and a collinear variable third real point is a uniquely defined torus generated by revolving a circle about an axis which cuts it in two real points. As the variable third point runs along the segments of the axis exterior to the two fixed points, the representatives of the corresponding Hessian points describe the outer sheet of the surface. As the third point runs along the segment between the two fixed points, the Hessian representatives describe the inner sheet of the surface. When the third point has the positions of the Marie representatives of the Hessian points for the two fixed points and the point at infinity on the axis, the Laguerre representatives of the corresponding Hessian points describe the locus of parabolic points on the surface; and when the third point runs along the segments of the axis exterior to these Marie representatives, the Laguerre representatives of the corresponding Hessian points describe the synclastic part of the outer sheet of the surface.*

The Jacobian points with respect to

$$I_1: (0, 1, -1), \quad I_2: (-1, 0, 1), \quad I_3: (a, -1, 1-a)$$

are

$$J_1: (2a, -1, 1-2a), \quad J_2: (a/2, -1, 1-a/2), \quad J_3: (-a, -1, 1+a).$$

It follows, because of the known mutual relations of these two triads of points, that the Hessian representatives for both the I and J points form the same two circles in the plane and the same torus in three-space.

6. *Circles as Representatives of Chains of Inflexions.* With

$$I_1: (0, 1, -1), \quad I_2: (-1, 0, 1), \quad I_3: (a, -1, 1-a)$$

as the real points of inflexion and $\Sigma(1, 1, 1)$ as the critic center common to the three real harmonic polars, the equations of the real harmonic polars are

$$h_1 \equiv \Sigma J_1 \equiv 3(x + 2ay) - (1 + 2a)(x + y + z) = 0,$$
$$h_2 \equiv \Sigma J_2 \equiv 3(2x + ay) - (2 + a)(x + y + z) = 0,$$
$$h_3 \equiv \Sigma J_3 \equiv 3(x - ay) - (1 - a)(x + y + z) = 0.$$

The Hessian lines of these harmonic polars are

$$3(x - \omega^i ay) - (1 - \omega^i a)(x + y + z) = 0, \quad (i = 1, 2),$$

and the sides of the real inflexion triangle

$$3x-(a+2+3\beta)(x+y+z)=0,$$
$$3ay-(1+2a+3\beta)(x+y+z)=0,$$
$$x+ay+\beta(x+y+z)=0.$$

The three pairs of imaginary inflexions are

$$\begin{array}{lll}
(3\beta\omega^2a+\omega^2a^2+2\omega^2a, & 3\beta-\omega a+1, & -3\beta\omega^2a-3\beta-\omega^2a^2-a-1),\\
(3\beta\omega a+\omega a^2+2\omega a, & 3\beta-\omega^2a+1, & -3\beta\omega a-3\beta-\omega a^2-a-1);\\
(3\beta\omega^2a+\omega^2a^2-\omega a, & 3\beta+2a+1, & -3\beta\omega^2a-3\beta-\omega^2a^2-\omega^2a-1),\\
(3\beta\omega a+\omega a^2-\omega^2a, & 3\beta+2a+1, & -3\beta\omega a-3\beta-\omega a^2-\omega a-1);\\
(3\beta\omega^2a+\omega^2a^2-a, & 3\beta-\omega^2a+1, & -3\beta\omega^2a-3\beta-\omega^2a^2+2\omega a-1),\\
(3\beta\omega a+\omega a^2-a, & 3\beta-\omega a+1, & -3\beta\omega a-3\beta-\omega a^2+2\omega^2a-1).
\end{array}$$

The real representatives for the pairs of imaginary inflexions are

$$\begin{aligned}
x&=-9\beta^2a(2a-1\pm3^{\frac12})-3a[4a^2+2a-3\pm2a(3^{\frac12})]\\
&\qquad-a[2a^3+3a^2-3a-2\pm(a^2-1)3^{\frac12}],\\
y&=9\beta^2[a-2\pm a(3^{\frac12})]-3\beta[2a+4\pm a(2a+3)3^{\frac12}]\\
&\qquad-a^3-3a-2\mp a(a+3a+2)3^{\frac12},\\
z&=18\beta^2(a^2-a+1)+6\beta(2a^3-2a^2+a+2)+2(a^4-a^3+2a+1);
\end{aligned}$$

$$\begin{aligned}
x&=-9\beta^2a(2a-1\pm3^{\frac12})-3\beta a[4a^2+2a\pm(3a+2)3^{\frac12}]\\
&\qquad-a[2a^3+3a^2+1\pm(2a^2+3a+1)3^{\frac12}],\\
y&=9\beta^2[a-2-a(3^{\frac12})]+3\beta[3a^2-2a-4-2a(3^{\frac12})]\\
&\qquad+2a^3+3a^2-3a-2\mp a(a^2-1)3^{\frac12},\\
z&=18\beta^2(a^2-a+1)+6\beta(2a^3+a^2-2a+2)+2(a^4+2a^3-a+1);
\end{aligned}$$

$$\begin{aligned}
x&=-9\beta^2a(2a-1\pm3^{\frac12})-3\beta a[4a^2+2a-3\pm(a+4)3^{\frac12}]\\
&\qquad-a[2a^3+3a^2-3a-2\pm3(a+1)3^{\frac12}],\\
y&=9\beta^2[a-2\mp a(3^{\frac12})]+3\beta[3a^2-2a-4\mp a(4a+1)3^{\frac12}]\\
&\qquad+2a^3+3a^2-3a-2\mp3a^2(a+1)3^{\frac12},\\
z&=18\beta^2(a^2-a+1)-6\beta(2a^3+a^2+a+2)\\
&\qquad+2(a^4+2a^3+3a^2+2a+1).
\end{aligned}$$

As β takes all values the representative point-pairs describe the following pairs of circles

$$\begin{aligned}
&2az(x-2y+z)+[\mp3^{\frac12}(x^2+y^2)\\
&\qquad-(2\mp3^{\frac12})xz+yz+(1\pm3^{\frac12})z^2]=0,\\
&2z(-2x+y+z)+a[\mp3^{\frac12}(x^2+y^2)+xz\\
&\qquad-(2\mp3^{\frac12})yz+(1\pm3^{\frac12})z^2]=0,\\
&\pm3^{\frac12}(x^2+y^2)-(2\pm3^{\frac12})xz+yz+(1\mp3^{\frac12})z^2\\
&\qquad-a[+3^{\frac12}(x^2+y^2)+xz-(2\pm3^{\frac12})yz+(1\mp3^{\frac12})z^2]=0.
\end{aligned}$$

These circles all pass through Σ, the first of each pair through $H^{(1)}$ and the second of each pair through $H^{(2)}$.

The addition of the equations of the mates of the pairs of circles gives the equations of the real harmonic polars; and shows that the second real intersections of the mates are respectively on h_1, h_2, h_3.

Symbolically the equations of the three pairs of circles may be written as

$$\begin{aligned} 2az\Sigma I_2 + K_1 &= 0, & 2az\Sigma I_2 + K_2 &= 0; \\ 2z\Sigma I_1 + aK_3 &= 0, & 2z\Sigma I_1 + aK_4 &= 0; \\ K_2 - aK_4 &= 0, & K_1 - aK_3 &= 0. \end{aligned}$$

The lines $\Sigma I_2, \Sigma I_1$ cut the Hessian circles $H_c^{(1)}, H_c^{(2)}$, respectively, in pairs of points

$$\begin{aligned} A^{(1)}&[-3-(2)3^{\frac{1}{2}},\ 1-3^{\frac{1}{2}},\ 5], & A^{(2)}&[-3+2(3^{\frac{1}{2}}),\ 1+3^{\frac{1}{2}},\ 5]; \\ B^{(1)}&[1-3^{\frac{1}{2}},\ -3-2(3^{\frac{1}{2}}),\ 5], & B^{(2)}&[1+3^{\frac{1}{2}},\ -3+2(3^{\frac{1}{2}}),\ 5]; \end{aligned}$$

and the circles K_1, K_2, K_3, K_4 pass respectively through $A^{(1)}, A^{(2)}, B^{(1)}, B^{(2)}$. The circles K_2, K_4 intersect in Σ and

$$C^{(1)}[1-3^{\frac{1}{2}},\ 1-3^{\frac{1}{2}},\ +2(3^{\frac{1}{2}})];$$

the circles K_1, K_3 intersect in Σ and

$$C^{(2)}[1+3^{\frac{1}{2}},\ 1+3^{\frac{1}{2}},\ -2(3^{\frac{1}{2}})];$$

where $C^{(1)}$ and $C^{(2)}$ are the centers of the circles $H_c^{(1)}, H_c^{(2)}$.

Hence *the Laguerre representation of the imaginary points of inflexion for the doubly infinite system of plane cubic curves with three real points of inflexion I_1, I_2, I_3 and a fixed critic center Σ, the one common to the three real harmonic polars h_1, h_2, h_3, consists of three uniquely defined pairs of circles*

$$\Sigma H^{(1)} A^{(1)},\ \Sigma H^{(2)} A^{(2)};\ \Sigma H^{(1)} B^{(1)},\ \Sigma H^{(2)} B^{(2)};\ \Sigma H^{(1)} C^{(1)},\ \Sigma H^{(2)} C^{(2)};$$

and the mates of these pairs have their second real intersections on h_1, h_2, h_3, respectively.

7. *Pencils of Representative Circles.* The forms

$$\begin{aligned} 2az\Sigma I_2 + K_1 &= 0, & 2az\Sigma I_2 + K_2 &= 0, \\ 2z\Sigma I_1 + aK_3 &= 0, & 2z\Sigma I_1 + aK_4 &= 0; \\ K_2 - aK_4 &= 0, & K_1 - aK_3 &= 0; \end{aligned}$$

show that as a varies the equations represent pairs of pencils of circles.

Hence the Laguerre representatives of the imaginary inflexions as I_3 runs along the line $I_1 I_2$ are pairs of pencils of circles on the points

$$\Sigma, A^{(1)} \text{ and } \Sigma, A^{(2)}; \quad \Sigma, B^{(1)} \text{ and } \Sigma, B^{(2)}; \quad \Sigma, C^{(1)} \text{ and } \Sigma, C^{(2)}.$$

8. *Pencils of Representative Critic Centers.* Methods similar to those used for the inflexions give that the imaginary critic centers for the cubics considered, in addition to the Hessian points, are represented by the pairs of pencils of circles

$$\begin{aligned}
&\mp 3^{\frac{1}{2}}(x^2+y^2) - (2 \mp 3^{\frac{1}{2}})xz + yz + (1 \pm 3^{\frac{1}{2}})z^2 \\
&\qquad + 2a[\pm 3^{\frac{1}{2}}(x^2+y^2) + xz - (2 \pm 3^{\frac{1}{2}})yz + (1 \mp 3^{\frac{1}{2}})z^2] = 0, \\
&\mp 3^{\frac{1}{2}}(x^2+y^2) - (2 \mp 3^{\frac{1}{2}})xz + yz - (1 \pm 3^{\frac{1}{2}})z^2 \\
&\qquad + (a/2)[\pm 3^{\frac{1}{2}}(x^2+y^2) + xz - (2 \pm 3^{\frac{1}{2}})yz + (1 \mp 3^{\frac{1}{2}})z^2] = 0, \\
&\mp 3^{\frac{1}{2}}(x^2+y^2) - (2 \mp 3^{\frac{1}{2}})xz + yz - (1 \pm 3^{\frac{1}{2}})z^2 \\
&\qquad - a[\pm 3^{\frac{1}{2}}(x^2+y^2) + xz - (2 \pm 3^{\frac{1}{2}})yz + (1 \mp 3^{\frac{1}{2}})z^2] = 0;
\end{aligned}$$

that is,

$$\begin{aligned}
&K_1 - 2aK_4 = 0, \qquad K_2 - 2aK_3 = 0; \\
&K_1 - (a/2)K_4 = 0, \quad K_2 - (a/2)K_3 = 0; \\
&K_1 - aK_4 = 0, \qquad K_2 - aK_3 = 0.
\end{aligned}$$

These circles are all through Σ; the first circle of each pair is through $M^{(1)}$ and the second of each pair through $M^{(2)}$, where $M^{(1)}$ and $M^{(2)}$ are the Marie representatives of the Hessian points when I_3 is at infinity; and the mates of the pairs intersect respectively in J_1, J_2, J_3.

Hence the Laguerre representatives of the imaginary critic centers, as I_3 runs along the line $I_1 I_2$, are the two Hessian circles $H_c^{(1)}, H_c^{(2)}$ and pencils of circles on

$$\Sigma, M^{(1)} \text{ and } \Sigma, M^{(2)},$$

with the second real intersections of the mates of the pairs at the Jacobian points of I_1, I_2, I_3.

WEST VIRGINIA UNIVERSITY.

A Character Symbol for Primes Relative to a Cubic Field.

By Chester G. Jaeger.

1. *Introduction.* Any cubic field defined by a binomial equation may have the defining equation written in the form

$$x^3 - ab^2 = 0$$

where a and b are relatively prime, and neither has a perfect square of a rational prime as a factor. In his study of the pure cubic field Dedekind * separates the types of fields into two kinds, I and II, according to whether $a^2 - b^2$ is not or is congruent to zero, mod 9. For I let $k = ab$, and for II let $k = 3ab$.

It has been shown that the class number, h, of a field is determined by the equation

$$gh = \lim_{s \to 1} (s-1)\zeta_K(s)$$

where g is a constant dependent on the field, and $\zeta_K(s)$ is Dedekind's zeta function for a field.

In order to determine the class number in the pure cubic field, Dedekind studied the factorization of rational primes in the quadratic field $K(\sqrt{-3})$; and from this he defined a function $\Psi(\mathfrak{p})$ which is the character of the prime ideals, as follows:

If $\mathfrak{p}$ is a factor of k, $\Psi(\mathfrak{p}) = 0$.
If $\mathfrak{p}$ is a factor of 3, but not of k, $\Psi(\mathfrak{p}) = 1$.
For all other prime ideals, $\Psi(\mathfrak{p}) = (ab^2/\mathfrak{p})_3$,

where $(ab^2/\mathfrak{p})_3 = \rho^i$, $(i = 1, 2, 3)$, and where ρ is a primitive cube root of unity. This symbol is the cubic residue character of ab^2 with respect to the modulus $\mathfrak{p}$. He then shows that we may write

$$\zeta_K(s) = \prod_p F(p),$$

where the product extends over all the rational primes, and

$$F(p) = [1/(1 - 1/p^s)] \cdot \prod 1/[1 - \Psi(\mathfrak{p})/N(\mathfrak{p})],$$

the product extending over the prime ideal factors of p.

* Dedekind, *Crelle's Journal*, Vol. 121, p. 40.

It is the object of this paper to define a function similar to $\Psi(\mathfrak{p})$ for a general cubic field. This will be accomplished in the following steps:

(1) It will be determined what values Ψ must have when the various types of primes, p, are factored in a certain quadratic field.

(2) The general cubic equation, $F_3(x) = 0$, will be transformed into a binomial cubic, $y^3 - \beta = 0$.

(3) Letting Δ_3 be the discriminant of $F_3(x) = 0$, and Δ_2 the discriminant of $F_2(x) = 0$, a quadratic equation related to $F_3(x) = 0$, the cubic character of β will be determined in the fields, $K(\Delta_2^{1/2})$ and $K(\omega, \Delta_2^{1/2})$. Here ω is a primitive cube root of unity.

(4) β is a cubic residue, mod p, if $y^3 - \beta$ has a solution in $K(p, \Delta_2^{1/2})$. This property is indicated by $\{\beta/\pi\} = \omega^i$, $i = 0, 1, 2$, depending on the type of prime. π is a factor of p in $K(\Delta_3^{1/2})$.

(5) If $\Psi(\pi) = \{\beta/\pi\}$, then, it will be shown that

$$\zeta_K(s) = \prod_p [1/(1 - 1/p^s)] \cdot \prod 1/[1 - \Psi(\mathfrak{p})/N(\mathfrak{p})],$$

the first product extending over all rational primes, and the second product over all the prime ideals of $K(\Delta_3^{1/2})$.

2. *Types of Primes.* Every rational prime p defines a principal ideal. The various ways in which this ideal may be resolved into its prime ideal factors in a cubic number field plays an important part in this article. A prime may be any one of five types—p_1, p_2, p_3, p_4, or p_5—according to the kinds of factors it has. Following is a table showing this, together with the norms of the factors.

Type	*Factors*	*Norms*
p_1	$\mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3$	$N(\mathfrak{p}_1) = N(\mathfrak{p}_2) = N(\mathfrak{p}_3) = p_1$
p_2	$\mathfrak{p}_1\mathfrak{p}_2$	$N(\mathfrak{p}_1) = p_2$, $\quad N(\mathfrak{p}_2) = p_2^2$
p_3	$\mathfrak{p}_1$	$N(\mathfrak{p}_1) = p_3^3$
p_4	$\mathfrak{p}_1\mathfrak{p}_2^2$	$N(\mathfrak{p}_1) = N(\mathfrak{p}_2) = p_4$
p_5	$\mathfrak{p}_1^3$	$N(\mathfrak{p}_1) = p_5$.

The factors of the primes of the types p_1, p_4, and p_5 are of the first degree. For the type p_2, $\mathfrak{p}_1$ is of the first degree, and $\mathfrak{p}_2$ is of degree 2. For the type p_3, $\mathfrak{p}_1$ is of degree 3. The notation, p_i, just introduced shall be adhered to throughout this article so that the subscript shall indicate the type of rational prime.

3. *The Function $\Psi(\pi)$.* In a quadratic field there are only three types

of factorization of a rational prime, p. That is, p is the product of two distinct primes, $p = \pi' \cdot \pi''$; or p is the square of a prime, $p = \pi^2$; or p remains prime, $p = \pi$. We shall associate with the general cubic field a certain quadratic field, Q, and then define a function, $\Psi(\pi)$, of all the prime ideals in Q such that

$$(1) \qquad \lim_{s \to 1} (s-1)\zeta_K(s) = \prod_{\pi} 1/[1 - \Psi(\pi)/N(\pi)],$$

π ranging over all the prime ideals of Q.

For any field, $\zeta_K(s)$ is made up of the product of all the fractions of the type

$$1/[1 - 1/N(\mathfrak{p})^s]$$

where $\mathfrak{p}$ ranges over all the prime ideals of the field. By grouping those ideals which arise from each of the types of the rational primes, p_i, and using the tabulated values of the norms in each case together with the fact that $(s-1)$ times the Riemann zeta function approaches unity as s approaches 1, we may write, for the cubic field,

$$(2) \qquad \lim_{s \to 1} (s-1)\zeta_K(s) = \prod_{p_1} [1/(1 - 1/p_1)]^2 \cdot \prod_{p_2} [1/(1 - 1/p_2^2)] \times \prod_{p_3} [1/(1 + 1/p_3 + 1/p_3^2)] \cdot \prod_{p_4} [1/(1 - 1/p_4)].$$

Thus, we see, by referring to (1) that Q must be so chosen, and $\Psi(\pi)$ so defined that the following must hold. (Here the norms are in Q).

For a prime of the type p_1,

$N(\pi') = N(\pi'') = p_1$, in which case $p_1 = \pi' \cdot \pi''$ in Q and $\Psi(\pi') = \Psi(\pi'') = 1$.

For a prime of the type p_2,

$N(\pi) = p_2^2$, in which case $p_2 = \pi$ in Q, and $\Psi(\pi) = 1$; or,
$N(\pi') = N(\pi'') = p_2$, in which case $p_2 = \pi' \cdot \pi''$ in Q, and $\Psi(\pi') = -\Psi(\pi'') = \pm 1$.

(We shall see later, § 5, that this second factorization of p_2 does not occur).

For a prime of the type p_3,

$N(\pi') = N(\pi'') = p_3$, in which case $p_3 = \pi' \cdot \pi''$ in Q and $\Psi(\pi') = \omega$, $\Psi(\pi'') = \omega^2$.

For a prime of the type p_4,

$N(\pi) = p_4$, in which case $p_4 = \pi^2$ in Q, and $\Psi(\pi) = 1$.

For a prime of the type p_5,

$N(\pi) = p_5$, or $N(\pi') = N(\pi'') = p_5$.

In either case $\Psi(\pi) = 0$, or $\Psi(\pi') = \Psi(\pi'') = 0$.

4. *The General Cubic Equation whose Roots Determine the Cubic Field.* There is no loss in generality in considering the general cubic equation to be of the form

$$x^3 + Cx + D = 0$$

since by linear transformation every cubic can be so expressed. Let us call this cubic $F_3(x)$. Now if C contains a power of p greater than or equal to 2, and if D contains a power of p greater than or equal to 3, then the roots of $F_3(x) = 0$ could be divided by a power of p, thus making a further reduction of the equation. We shall therefore assume that either p occurs to a power less than 2 in C or to a power less than 3 in D.

Closely associated with $F_3(x)$ is the quadratic

$$F_2(x) = 3Cx^2 + 9Dx - C^2 = 0.$$

If we let Δ_3 and Δ_2 be, respectively, the discriminants of $F_3(x) = 0$ and $F_2(x) = 0$, then it is easily verified that

$$\Delta_2 = -3\Delta_3.$$

Call the roots of $F_2(x) = 0$, μ_1 and μ_2. Then, if we apply the transformation

$$x = (\mu_1 y + 3\mu_1\mu_2)/(y + 3\mu_1) \tag{3}$$

to $F_3(x) = 0$, it becomes the binomial cubic

$$y^3 - 9C\mu_1 = 0.$$

And, finally, for convenience of notation, if we let $\beta = 9C\mu_1$, we have

$$y^3 - \beta = 0, \tag{4}$$

which has its coefficients in $K(\Delta_2^{1/2})$.

5. *The Quadratic Character of* Δ_3, mod p. In the field $K(p_1)$, $F_3(x)$ has three linear factors.

$$F_3(x) = (x - a_1)(x - a_2)(x - a_3) \qquad (p_1)$$

Obviously,

$$\Delta_3 = (a_1 - a_2)^2(a_1 - a_3)^2(a_2 - a_3)^2 \qquad (p_1)$$

is a perfect square. From this it follows that $(\Delta_3/p_1) = 1$. Hence, in $K(\Delta_3^{1/2})$, $p_1 = \pi' \cdot \pi''$.

In the field $K(p_2)$, $F_3(x)$ is factorable into a linear and a quadratic factor,

$$F_3(x) = (x - a)(x^2 - bx + c) \qquad (p_2).$$

In this case

$$\begin{aligned}\Delta_3 &= D(x-a)D(x^2-bx+c)R[(x-a),\ (x^2-bx+c)]^2 \\ &= D(x^2-bx+c)R[(x-a),\ (x^2-bx+c)]^2.\end{aligned}$$

Hence, we see that the discriminant of $F_3(x)$ differs from the discriminant of $x^2 - bx + c$ only by a square factor. But this means that the quadratic field $K(\Delta_3^{1/2})$ must be identical with the quadratic field determined by the roots of the equation

$$x^2 - bx + c = 0. \qquad (p_2)$$

This last equation is irreducible in $K(p_2)$ and hence its discriminant could not be a quadratic residue module p_2.* Therefore $(\Delta_3/p_2) = -1$, or, in $K(\Delta_3^{1/2})$, $p_2 = \pi$. (Note that this excludes one of the possibilities of factoring p_2 mentioned in § 3).

In $K(p_3)$, $F_3(x)$ is irreducible. And for $p_3 > 3$, p_3 is relatively prime to Δ_3.† Let α be a number satisfying the congruence

$$F_3(x) \equiv 0, \text{ mod } p_3.$$

Then, since p_3 is relatively prime to Δ_3, $F_3(x)$ is a prime function, mod p_3. Therefore,

$$F_3(x) \equiv (x - \alpha)(x - \alpha^{p_3})(x - \alpha^{p_3^2}), \text{ mod } p_3; \text{ or}$$
$$\Delta_3 \equiv \delta^2, \text{ mod } p_3, \text{ where } \delta = (\alpha - \alpha^{p_3})(\alpha - \alpha^{p_3^2})(\alpha^{p_3} - \alpha^{p_3^2}).$$

δ is rational, mod p_3, since it is unchanged by the substitution, (α, α^{p_3}). Hence Δ_3 is a quadratic residue, mod p_3. So, $(\Delta_3/p_3) = 1$; and in $K(\Delta_3^{1/2})$, $p_3 = \pi' \cdot \pi''$.

We shall now apply the theorem of Dedekind to

$$p_4 = \mathfrak{p}_1 \mathfrak{p}_2^2.$$

Since we are considering only $p_4 > 3$, the discriminant of $K(\alpha)$‡ is divisible by p_4 with the exponent 1. It is a well known fact that the discriminant of

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* Wahlin, *American Journal of Mathematics*, Vol. 44. If p is contained in Δ_3 it is to an even power, and hence $(\Delta_3/p_2) = (\Delta_3'/p_2)$, where $\Delta_3 = p_2^s \Delta_3'$.

† Wahlin, *loc. cit.* Since p_3 does not occur when $\Delta_3 \equiv 0$, mod p_3.

‡ α is a root of $F_3(x) = 0$.

an equation differs from the field discriminant by a square factor. Therefore Δ_3 is divisible by p_4 an odd number of times, since this square factor, if it contains p_4 at all must contain it an even number of times. Now in $K(p_4)$, $F_3(x)$ has a quadratic and a linear factor,

$$F(x) = (x - a)(x^2 - bx + c) \qquad (p_4).$$

As in the case of p_2, we could show that

$$\Delta_3 = A^2 D(x^2 - bx + c).$$

Now since Δ_3 contains p_4 an odd number of times, $D(x^2 - bx + c)$ must contain it an odd number of times, and therefore at least once. Thus the discriminant of $K(\Delta_3^{1/2})$ is divisible by p_4. So, in $K(\Delta_3^{1/2})$, $p_4 = \pi^2$.

6. *The Cubic Character of β in $K(\Delta_2^{1/2})$ with respect to π_1 and π_2.* If

$$F_3(x) = (x - r_1)(x - r_2)(x - r_3), \qquad (p)$$

then by applying the transformation (3) we have

$$y^3 - \beta = (y - \rho_1)(y - \rho_2)(y - \rho_3) \qquad (p) \tag{5}$$

where ρ_1, ρ_2 and ρ_3 are numbers in the Ring $R(p, \Delta_2^{1/2})$.

And, if

$$F_3(x) = (x - r)(x^2 + mx + n) \qquad (p)$$

then the transformation gives

$$y^3 - \beta = (y - \rho)(y^2 + \mu y + \nu) \qquad (p) \tag{6}$$

where ρ, μ and ν are numbers in the Ring $R(p, \Delta_2^{1/2})$.

Equations (5) and (6) are possible when and only when β is a cubic residue mod p in the field $K(\Delta_2^{1/2})$. And by Euler's Criterion this is true when and only when

$$\beta^{[N(\pi_1)-1]/\delta} \equiv 1, \bmod \pi_1,$$

where δ is the greatest common division of $N(\pi_1) - 1$ and 3, and where π_1 is any prime factor of p in $K(\Delta_2^{1/2})$. Now β may contain π_1,

$$\{\beta\} = \pi_1^k \bar{\bar{\beta}}$$

$\bar{\bar{\beta}}$ being prime to π_1.

If $k \equiv 0 \bmod 3$ and π_1 is a principal ideal, then dividing the roots of (4) by $\pi_1^{k/3}$ would give

$$y^3 - \bar{\beta} = 0.$$

If $k \equiv 0 \bmod 3$ and π_1 is not a principal ideal, then the principal ideal $\{\beta\}$ is also divisible by π_1^k. Now there exists an ideal I, such that I and $\{\beta\}$ are relatively prime, and such that $I \cdot \pi_1$, is a principal ideal, $\{\gamma\}$. Since I and $\{\beta\}$ are relatively prime there is an integer in I which is prime to β. Call this integer λ. Now, if we multiply the roots of (4) by $\lambda^{k/3}$ and divide them by $\gamma^{k/3}$ we get

$$y^3 - \bar{\beta} = 0.$$

If $k \not\equiv 0 \bmod 3$, then it can be shown that equation (4) is irreducible, and that further π_1 is the cube of a prime ideal.*

Let β' be the conjugate of β. A short calculation shows that

$$\beta\beta' = -27C^3.$$

Thus, if β is a cubic residue mod π_1, β' is also a cubic residue mod π_1; and β' is a cubic residue with respect to the conjugate ideal π_2. Hence, from the relation existing between β and β', we can conclude that if β' is a cubic residue mod π_1, it is also a cubic residue mod π_2. (Of course, if p remains prime in $K(\Delta_2^{1/3})$, then this last consideration is superfluous).

7. *The Cubic Character of β in the Field $K(\omega, \Delta_2^{1/3})$ with Respect to π' and π''.* Since $\Delta_2 = -3\Delta_3$, we see immediately that the composite field $K(\Delta_3^{1/2}, \Delta_2^{1/3})$ is the same as the field $K(\omega, \Delta_2^{1/3})$. We shall show that β is a cubic residue mod π' and π'' if it is a cubic residue mod π_1 and π_2. This will depend on the ways in which p is factorable in the two fields $K(\Delta_2^{1/3})$ and $K(\Delta_3^{1/2})$. Consider the four possibilities (A), (B), (C), and (D):

(A) In $K(\Delta_2^{1/3})$, $p = \pi_1 \cdot \pi_2$
In $K(\Delta_3^{1/2})$, $p = \pi' \cdot \pi''$.

We saw that in this case (§ 5) Δ_3 is a quadratic residue in the rational field mod p, and hence certainly mod π_1 and mod π_2 in the field $K(\Delta_2^{1/3})$. Thus, in the field $K(\omega, \Delta_2^{1/3})$, π_1 and π_2 are each factorable into two distinct factors. $\pi_1 = \pi_1' \cdot \pi_1''$, $\pi_2 = \pi_2' \cdot \pi_2''$; and from unique factorization $\pi' = \pi_1'\pi_2'$ and $\pi'' = \pi_1'' \cdot \pi_2''$ in the field $K(\omega, \Delta_2^{1/3})$. Thus, if β is a cubic residue mod π_1 and π_2 in $K(\omega, \Delta_2^{1/3})$, then it is certainly a cubic residue mod π_1', π_2', π_1'' and π_2''. But this says it is a cubic residue mod π' and mod π'' in $K(\omega, \Delta_2^{1/3})$. We shall indicate this cubic residue property by

$$\{\beta/\pi'\} = \{\beta/\pi''\} = +1.$$

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* Wahlin, *Crelle's Journal*, Vol. 145, p. 133.

(B) In $K(\Delta_2^{1/2})$, $p = \pi_1 \cdot \pi_2$
In $K(\Delta_3^{1/2})$, $p = \pi'$.

In this case, we saw (§ 5) that Δ_3 is not a quadratic residue mod $\pi' = p$. And since $\Delta_2 = -3\Delta_3$ is a quadratic residue, -3 must be a quadratic non-residue mod p. Hence

$$p + 1 \equiv 0, \text{ mod } 3.$$

In this case, then, $\delta = 1$. By hypothesis

$$\beta^{p-1} \equiv 1, \text{ mod } \pi_1 \text{ and } \pi_2.$$

Then surely

$$\beta^{p-1} \equiv 1, \text{ mod } \pi'.$$

$(p+1)/3$ is an integer. Raising both sides to the power $(p+1)/3$ we have

$$\beta^{(p^2-1)/3} \equiv 1, \text{ mod } \pi'.$$

Now, since $N(\pi')^* = p^2$, this becomes

$$\beta^{[N(\pi')-1]/3} \equiv 1, \text{ mod } \pi' \text{ or}$$
$$\{\beta/\pi'\} = 1$$

(C) In $K(\Delta_2^{1/2})$, $p = \pi_1$
In $K(\Delta_3^{1/2})$, $p = \pi' \cdot \pi''$.

We see at once that $N(\pi_1) = p^2$. And since

$$\beta^{[N(\pi_1)-1]/3} = \beta^{(p^2-1)/3} \equiv 1, \text{ mod } \pi_1,$$

it follows that

$$\beta^{(p^2-1)/3} \equiv 1, \text{ mod } \pi' \text{ and } \pi''.$$

This says that in $K(\omega, \Delta_2^{1/2})$, β is cubic residue mod π' and π''. Or,

$$\{\beta/\pi'\} = \{\beta/\pi''\} = 1.$$

(D) In $K(\Delta_2^{1/2})$, $p = \pi_1$,
In $K(\Delta_3^{1/2})$, $p = \pi'$.

Here $(\Delta_3/p) = -1$, and $(\Delta_2/p) = -1$. Thus $(\Delta_2\Delta_3/p) = +1 = (-3\Delta_3^2/p) = (-3/p)$. So in $K(\omega, \Delta_2^{1/2})$, p may have two factors $\pi^{(1)}$, $\pi^{(2)}$. Then since $\beta^{(p^2-1)/3} \equiv 1$, mod π_1, it follows that in the field $K(\omega, \Delta_2^{1/2})$, $\beta^{(p^2-1)/3} \equiv 1$, mod $\pi^{(1)}$ and $\pi^{(2)}$, and hence, certainly

$$\beta^{(p^2-1)/3} \equiv 1, \text{ mod } \pi'.$$

Or, we may write

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* It is understood in all cases, norm is taken in the field $K(\Delta_3^{1/2})$.

$$\beta^{[N(\pi)-1]/3} \equiv 1, \text{ mod } \pi'.$$

Thus, for this case also we have

$$\{\beta/\pi\} = 1.$$

8. *The Function* $\Psi(\pi)$. If in $K(\Delta_2^{1/2})$ $p = \pi_1 \cdot \pi_2$ and β is a cubic residue mod π_1 and mod π_2 it follows that it is a cubic residue mod p, provided p does not happen to be a factor of Δ_2,

$$\Delta_2 = p^s \Delta_2'$$

with s odd. For p larger than 3, p would be contained in Δ_3 to the same power as in Δ_2. From the theorem of Dedekind we see that primes of the types p_1 and p_2 are not contained in the field discriminant of $K(\alpha)$, and hence if in Δ_3, it must be to an even power.

We also saw that in $K(\Delta_3^{1/2})$, $p_1 = \pi' \cdot \pi''$. So this fits the case (A) just considered. That is, for primes of the type p_1,

$$\{\beta/\pi'\} = \{\beta/\pi''\} = 1.$$

Let us put $\Psi(\pi) = \{\beta/\pi\}$. Then, since $N(\pi') = N(\pi'') = p_1$, we have

$$1/(1 - 1/p_1)^2 = \prod_{i=1}^{2} 1/[1 - \Psi(\pi^{(i)})/N(\pi^i)].$$

And in $K(\Delta_3^{1/2})$, $p_2 = \pi'$. This is the case, (B) or (D), where $\{\beta/\pi'\}=1$. Here, $N(\pi') = p_2^2$. So

$$1/(1 - 1/p_2^2) = 1[1 - \Psi(\pi')/N(\pi')].$$

In § 6, equations (5) and (6) are the cases which arose from $F_3(x)$ being factorable in the ring $R(p)$. Let us next take up the case where $F_3(x)$ is irreducible in the ring $R(p)$. This is the case where p is of the type p_3. In this case $F_3(x)$ can not be resolved into factors involving only quadratic irrationalities. Thus $y^3 - \beta$ must also be irreducible in the ring $R(p, \Delta_2^{1/2})$, and β could not be a cubic residue. For, if $y^3 - \beta$ were factorable in $R(p, \Delta_2^{1/2})$, $y^3 - \beta = 0$ would have at least one root in this ring. Then $F_3(x)$ would have a root in $R(p, \Delta_2^{1/2})$. But, since $F_3(x)$ is irreducible in $R(p)$, it cannot have factors involving only quadratic irrationalities.

Thus for primes of the type p_3, β is not a cubic residue. In order to find the function $\Psi(\pi)$ we have two cases to consider: (a) when $p_3 = \pi_1 \cdot \pi_2$ in $K(\Delta_2^{1/2})$ and (b) when $p_3 = \pi_1$ in $K(\Delta_2^{1/2})$. In either case, we have seen, page ——, that $p_3 = \pi' \cdot \pi''$ in $K(\Delta_3^{1/2})$.

(a) This corresponds to (A) above. In this case we saw that $\pi_1 = \pi_1' \pi_1''$ and that $\pi_2 = \pi_2' \pi_2''$ in the field $K(\omega, \Delta_2^{1/2})$. By Fermat's theorem

$$\beta^{p_3-1} - 1 \equiv 0, \text{ mod } \pi_1 \text{ and } \pi_2$$

since π_1 and π_2 are primes of the first degree in $K(\Delta_2^{1/2})$. Moreover,

$$\beta^{(p_3-1)/3} \not\equiv 1, \text{ mod } \pi_1 \text{ or } \pi_2$$

since β is not a cubic residue mod π_1 or π_2. Thus, if we write

$$\beta^{p_3-1} - 1 = [\beta^{(p_3-1)/3} - 1][\beta^{(p_3-1)/3} - \omega]\,[\beta^{(p_3-1)/3} - \omega^2] \equiv 0, \text{ mod } \pi_1 \text{ and } \pi_2$$

it is obvious that

$$[\beta^{(p_3-1)/3} - \omega]\,[\beta^{(p_3-1)/3} - \omega^2] \equiv 0, \text{ mod } \pi_1 \text{ and } \pi_2.$$

Therefore, $\beta^{(p_3-1)/3} - \omega$ is divisible by a factor of π_1 or $\beta^{(p_3-1)/3} - \omega^2$ is divisible by π_1. But, since $\beta^{(p_3-1)/3}$ is congruent to a rational integer, b, mod π_1, if $\beta^{(p_3-1)/3} - \omega^2$ is divisible by π_1, $\omega^2 - b$ is divisible by π_1. However, ω^2 is the root of an irreducible equation of degree 2 relative to $K(\Delta_2^{1/2})$, and π_1 is a prime ideal in this field which is not a divisor of the index of ω^2. Hence, ω^2 cannot satisfy a congruence of degree less than 2 mod π_1, and thus

$$\beta^{(p_3-1)/3} - \omega^2 \equiv 0, \text{ mod } \pi_1$$

is impossible.

In the same way, $\beta^{(p_3-1)/3} - \omega$ is not divisible by π_1. But, $[\beta^{(p_3-1)/3} - \omega]\,[\beta^{(p_3-1)/3} - \omega^2]$ is divisible by π_1 and by π_2 and hence by the four factors π_1', π_2', π_1'' and π_2''. Two of these must be divisors of $\beta^{(p_3-1)/3} - \omega$ and two of $\beta^{(p_3-1)/3} - \omega^2$. Let us assume that the notation is such that π_1' is a divisor of $\beta^{(p_3-1)/3} - \omega$. Then the other divisor is π_2' or π_2''. First, assume it is π_2''. Then $\beta^{(p_3-1)/3} - \omega^2$ is divisible by π_2', and β is congruent to a rational integer, c, mod π_2'; it follows that

$$c - \omega^2 \equiv 0, \text{ mod } \pi_2'.$$

Considering $K(\omega, \Delta_2^{1/2})$ as a relative field to $K(\Delta_3^{1/2})$, it follows from a change of $-\Delta_2^{1/2}$ into $\Delta_2^{1/2}$ that $(-3)^{1/2} = \Delta_2^{1/2}/\Delta_3^{1/2}$ is transformed into $-(-3)^{1/2}$. Consequently, from $b - \omega \equiv 0$, mod π_1', follows $b - \omega^2 \equiv 0$, mod π_2'. Therefore

$$b \equiv c, \text{ mod } p_3.$$

Let β' be the conjugate of β. Then, from

$$b \equiv \beta^{(p_3-1)/3}, \text{ mod } \pi_1$$

we have

(7) $$b \equiv \beta'^{(p_3-1)/3}, \text{ mod } \pi_2;$$

also

(8) $$\beta^{(p_3-1)/3} \equiv c \equiv b, \text{ mod } \pi_2.$$

Thus, from (7) and (8) it follows that

(9) $$(\beta\beta')^{(p_3-1)/3} \equiv b^2, \text{ mod } \pi_2.$$

But $\beta\beta' = N(\beta) = -27C^3$, and hence

$$(\beta\beta')^{(p_3-1)/3} \equiv (-3C)^{p_3-1} \equiv 1, \text{ mod } \pi_2.$$

Then, from (8) and (9)

$$\beta^{2[(p_3-1)/3]} \equiv 1, \text{ mod } \pi_2 \quad \text{or} \quad \beta^{(p_3-1)/3} \equiv 1, \text{ mod } \pi_2,$$

which contradicts the assumption that $\beta^{(p_3-1)/3} \not\equiv 1$, mod π_2. Therefore, $\beta^{(p_3-1)/3} - \omega \equiv 0$, mod π_2'' is impossible. We must thus conclude that

$$\beta^{(p_3-1)/3} \equiv 0, \text{ mod } \pi_2'.$$

And finally, therefore,

$$\beta^{(p_3-1)/3} - \omega \equiv 0, \text{ mod } \pi_1'\pi_2' (= \pi'). \tag{10}$$

If p_3 is a divisor of the index of $\beta^{(p_3-1)/3}$ then

$$\beta^{(p_3-1)/3} \equiv \beta'^{(p_3-1)/3}, \text{ mod } p_3.$$

But then

$$\beta^{2[(p_3-1)/3]} \equiv (\beta\beta')^{(p_3-1)/3} \equiv 1, \text{ mod } p_3 \quad \text{or} \quad \beta^{(p_3-1)/3} \equiv 1, \text{ mod } p_3,$$

contrary to our assumption. Hence p_3 cannot be a divisor of the index of $\beta^{(p_3-1)/3}$.

So, for $p_3 = \pi_1 \cdot \pi_2$ in $K(\Delta_2^{1/2})$,

$$\Psi(\pi') = \omega, \quad \text{and} \quad \Psi(\pi'') = \omega^2.$$

And, since $N(\pi') = N(\pi'') = p_3$

$$[1/(1-\omega/p_3)]\,[1/(1-\omega^2/p_3)] = \prod_{i=1}^{2} \{1/[1-\Psi(\pi^{(i)})/N(\pi^{(i)})]\}. \tag{11}$$

(b) If in $K(\Delta_2^{1/2})$ p_3 remains prime, then by Fermat's theorem

$$\beta^{p_3^2-1} - 1 \equiv 0, \text{ mod } \pi_1,$$

since then π_1 is a prime of degree 2 in $K(\Delta_2^{1/2})$.

In a manner precisely similar to the case just treated, it is easily seen that

$$\beta^{(p_3^2-1)/3} \equiv \omega, \text{ mod } \pi' \quad \text{and} \quad \beta^{(p_3^2-1)/3} \equiv \omega^2, \text{ mod } \pi''.$$

Thus $\Psi(\pi') = \omega$, $\Psi(\pi'') = \omega^2$ and, in either case (a) or (b), for a prime of the type p_3, the equation (11) holds. Since for p_4

$$F_3(x) = (x-a)(x^2-bx+c), \qquad (p_4)$$
$$y^3 - \beta = (y-\rho)(y^2+\mu y+\nu) \qquad (p_4)$$

in $R(p_4, \Delta_2^{1/2})$ and hence β is a cubic residue.

$$\Psi(\pi') = \{\beta/\pi'\} = 1, \quad N(\pi') = p_4, \quad 1/(1-1/p_4) = 1/[1-\Psi(\pi)/N(\pi)].$$

We shall assign the value 0 to $\Psi(\pi')$, or $\Psi(\pi')$ and $\Psi(\pi'')$ for primes of the type p_5.

9. $\Psi(\pi)$ *for* 3 *and* 2. Nothing in this discussion has excluded the primes 3 and 2 except when they are of the types p_3 and p_4. Wahlin has shown * that when 3 is of the type p_3, then either $s = 0$, and $\Delta_3 \equiv 1$, mod 3 or $s = 6$ and $\Delta_3' \equiv 1$, mod 3. In either case, Δ_3 is a quadratic residue mod 3, and hence in $K(\Delta_3^{1/2})$, $3 = \pi' \cdot \pi''$.
So we shall define

$$\{\beta/\pi'\} = \omega, \qquad \{\beta/\pi''\} = \omega^2.$$

The only case where 2 remains prime in a cubic field is when Δ_3 is odd and $C \not\equiv 0 \bmod 2$. Now $\Delta_3 = -27D^2 - 4C^3$. The following congruence is then obvious:

$$\Delta_3 = -27D^2 - 4C^3 \equiv 1, \text{ mod } 8;$$

and hence, $2 = \pi' \cdot \pi''$; again we shall define

$$\{\beta/\pi'\} = \omega, \qquad \{\beta/\pi''\} = \omega^2.$$

When 3 is of the type p_4, from this same article by Wahlin, it can be shown that in all cases s is odd, and therefore $3 = \pi^2$. So, we shall, since $y^3 - \beta$ is reducible in $R(2, \Delta_2^{1/2})$, define

$$\{\beta/\pi\} = 1.$$

When 2 is of the type p_4, s is either odd, in which case $2 = \pi^2$; or else s is even in which case $2 = \pi'\pi''$. In either case, we shall set

$$\Psi(\pi') = 1 \quad \text{or} \quad \Psi(\pi') = \Psi(\pi'') = 1.$$

In § 6, we saw that if in $\beta = \pi_1{}^k\bar{\beta}$ $k \not\equiv 0$, mod 3, then π_1 is the cube of a prime ideal in $K(\beta^{1/3}, \Delta_2^{1/2})$. Now p_5 becomes $\mathfrak{p}_1{}^3$ in $K(\alpha)$. Then by Dedekind's theorem, we see that p_5 is contained an even number of times in Δ_3, and (for $p_5 > 3$) in Δ_2. Thus in $K(\Delta^{1/2})$, $p_5 = \pi_1$, or $p_5 = \pi_1 \cdot \pi_2$. So, in $K(\beta^{1/3}, \Delta_2^{1/2})$, $p_5 = \bar{\pi}_1{}^3$ or $\bar{\pi}_1{}^3\bar{\pi}_2{}^3$. That is, when $k \not\equiv 0$, mod 3, we have the case p_5. This is the only way in which p_5 occurs; and no other prime occurs in this way. So for all other primes, $k \equiv 0$, mod 3, and $\bar{\beta}$ has the same cubic character as β. Therefore,

If $k \equiv 0$, mod 3, $\{\beta/\pi\} = \{\bar{\beta}/\pi\} = \omega^i$, $(i = 0, 1, 2)$.
If $k \not\equiv 0$, mod 3, $\{\beta/\pi\} = 0$.

Finally, then, we have

$$\zeta_K(s) = \prod_p 1/(1 - 1/p^s) \cdot \prod_\pi 1/[1 - \Psi(\pi)/N(\pi)],$$

the first product extending over all rational primes, p, and the second product over all the prime ideals in $K(\Delta_3^{1/2})$.

* *American Journal of Mathematics*, Vol. 44, pp. 202 ff.

Differential Equations of Infinite Order with Constant Coefficients.*

By Harold T. Davis.

1. *Introduction.* In a fundamental paper F. Schürer † has discussed the solution of the differential equation of infinite order

$$a_0u(x) + a_1u'(x) + a_2u''(x) + \cdots = f(x), \tag{1}$$

in which the coefficients are assumed to be constants and $f(x)$ is an infinitely differentiable function subject to the condition

$$\lim_{n=\infty} |f^{(n)}(x)|^{1/n} \leqq L, \qquad (L \text{ finite}). \tag{2}$$

It is clear that equation (1) is of rather general application since its theory is closely associated with the theory of functional equations of the following types:

$$\text{(a)} \quad u(x) + \int_x^\infty \sum_{i=1}^n e^{s_i(x-t)}u(t)\,dt = f(x), \qquad s_i > 0,$$

$$\text{(b)} \quad u(x) + \int_a^b K(t)u(x+ct)\,dt = f(x),$$

$$\text{(c)} \quad u(x+b) + \lambda u(x) = f(x).$$

This relationship is formally exhibited by expanding $u(T)$ in a Taylors series about x,

$$u(T) = u(x) + (T-x)u'(x) + (T-x)^2u''(x)/2! + \cdots.$$

If we replace T by t and substitute in (a) we get a differential equation in which the coefficients are,

$$a_0 = 1 + \sum_{m=1}^n 1/s_m, \qquad a_i = \sum_{m=1}^n 1/s_m^{i+1}, \qquad i > 0.$$

Similarly, if we set $T = x + ct$, equation (b) is seen to reduce to a differential equation of type (1) with the coefficients

$$a_0 = 1 + \int_a^b K(t)\,dt, \qquad a_i = \int_a^b [K(t)(ct)^i/i!]\,dt, \qquad i > 0.$$

* Presented to the American Mathematical Society, August 27, 1929, at Boulder, Colorado.

† "Eine gemeinsame Methode zur Behandlung gewisser Funktional-gleichungsprobleme," *Leipziger Berichte*, Vol. 70 (1918), pp. 185-246.

Letting $T = x + b$, equation (c) reduces to a differential equation with coefficients

$$a_0 = 1 + \lambda, \qquad a_i = b^i/i!, \qquad i > 0.$$

One object of the present paper is to discuss the operational solution of equation (1) when the condition (2) imposed by Schürer fails to hold. A second object is to extend theorems of the Heaviside operational calculus to infinite operators by methods which have some claim to novelty since they exhibit a fundamental relationship connecting the Heaviside expansion theorem with the expansion of the operators in Laurent series.*

2. *Expansion of the Resolvant Generatrix in a Laurent Series.* By the generatrix of equation (1) we shall mean the function

$$F(z) = a_0 + a_1 z + a_2 z^2 + \cdots .$$

It is well known that the solution of (1) is given symbolically by the *resolvant generatrix*, $G(z) = 1/F(z)$, in the form

$$u(x) = G(z) \rightarrow f(x),$$

where we employ the symbol $G(z) \rightarrow f(x)$ to signify the function obtained by operating upon $f(x)$ by $G(z)$.†

We first prove the theorem:

THEOREM 1. *If $G_1(z)$ designates a Laurent expansion of $G(z)$ in an annulus formed by two concentric circles about the origin and if $G(z)$ is any other expansion about the origin, then the function*

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* Since the completion of this paper three studies relating to the same subject have appeared. In a memoir in the *Annales de l'École Normale Supérieure*, Ser. 3, Vol. 46 (1929), pp. 25-53, G. Valiron has investigated the nature of the solution of the homogeneous equation by means of methods originally due to J. F. Ritt. The case of linear systems of differential equations of infinite order with constant coefficients has been the subject of two memoirs by I. M. Sheffer. See the *Annals of Mathematics*, Ser. 2, Vol. 30 (1929), pp. 250-264; *Transactions of the American Mathematical Society*, Vol. 31 (1929), pp. 281-289. In these latter the degree (Stufe), (see section 3), is limited to the finite case and the approach, differing from that of the present paper, is made through consideration of an equivalent system of linear algebraic equations.

† This follows rigorously from Bourlet's generatrix equation,

$$[G \cdot F] = F \cdot G + \frac{\partial F}{\partial x}\frac{\partial G}{\partial z} + \frac{1}{2!}\frac{\partial^2 F}{\partial x^2}\frac{\partial^2 G}{\partial z^2} + \cdots = 1.$$

Since $F(z)$ is independent of x, $F \cdot G = 1$. See C. Bourlet, "Sur les opérations en général et les équations différentielles linéaires d'ordre infini," *Annales de l'école normale supérieure*, Ser. 3, Vol. 14 (1897), pp. 133-190.

$$U(x) = \{G_1(z) - G(z)\} \to f(x),$$

where $f(x)$ is arbitrary to within the limits of the existence of the right hand member, is a solution of the homogeneous equation

$$F(z) \to u(x) = 0.$$

In order to prove this theorem let us assume first that $F(z)$ is of the form

$$F(z) = (z - a)/\phi(z),$$

where $\phi(z)$ has no singularity within or upon the circle of radius $r = a$.

The inverse operator, $G(z) = \phi(z)/(z - a)$, then has the two expansions,

$$G(z) = -\{1 + (z/a) + (z/a)^2 + \cdots\}\,\phi(z)/a,$$
$$G_1(z) = \{1/z + a/z^2 + a^2/z^3 + \cdots\}\phi(z),$$

the first expansion being valid within the circle of radius a and the second in the region exterior to it.

We make the usual interpretation that $1/z \to f(x) = \int_0^x f(t)\,dt$. Replacing $f(t)$ by its Taylor's expansion we obtain

$$1/z \to f(x) = (1 - e^{-xz})/z \to f(x),$$

which expresses the integral as a differential operator.

Similarly it may be proved that

$$1/z^2 \to f(x) = \{1 - (1 + xz)e^{-xz}\}/z^2 \to f(x),$$

and, in general,

$$\begin{aligned} 1/z^n \to f(x) &= \{1 - [1 + xz + x^2z^2/2! \\ &\qquad + \cdots + x^{n-1}z^{n-1}/(n-1)!]e^{-xz}\}/z^n \to f(x), \\ &= \{x^n/n - x^{n+1}z/(n+1) \\ &\qquad + x^{n+2}z^2/2!(n+2) - \cdots\}/(n-1)! \to f(x). \end{aligned}$$

Letting $\phi(z) = 1$, and replacing $1/z^n$ in the expansion of $G_1(z) \to f(x)$ by the expression given above we get:

$$\begin{aligned} G_1(z) \to f(x) = \{&(1 - e^{-xz})/z + a[1 - (1 + xz)e^{-xz}]/z^2 + a^2[1 - (1 + xz \\ &+ x^2z^2/2!)e^{-xz}]/z^3 + \cdots\} \to f(x). \end{aligned}$$

It will be clear from the explicit form of $1/z^n \to f(x)$ that $G_1(0) = \sum_{n=1}^{\infty} a^n x^n/n!\,a = (e^{ax} - 1)/a$, $G_1'(0) = -\sum_{n=1}^{\infty} a^{n+1}x^{n+1}/(n-1)!\,(n+1)a^2 = -\{e^{ax}(ax - 1) + 1\}/a^2$, $G''(0) = \{e^{ax}(a^2x^2 - 2ax + 2) - 2\}/a^3$, and,

in general, $G_1^{(n)}(0)=(-1)^n\{e^{ax}[a^nx^n-na^{n-1}x^{n-1}+n(n-1)a^{n-2}x^{n-2}-\cdots \pm n!]-n!\}/a^{n+1}$.

If we replace these values in the expansion

$$G_1(z)=G_1(0)+G_1'(0)z+G_1''(0)z^2/2!+\cdots,$$

and then collect the coefficient of e^{ax}, we see that the operator reduces to the expression,

$$G_1(z)=e^{ax}e^{-ax}(1/a+z/a^2+z^2/a^3+\cdots)\\ -(1/a+z/a^2+z^2/a^3+\cdots).$$

Taking account of the fact that $z^ne^{-ax}\to f(x)=f^{(n)}(0)$, we get: *

$$G_1(z)\to f(x)=e^{ax}\{f(0)/a+f'(0)/a^2+f''(0)/a^3+\cdots\}\\ -\{f(x)/a+f'(x)/a^2+f''(x)/a^3\cdots\},$$
$$=e^{ax}\{f(0)/a+f'(0)/a^2+f''(0)/a^3+\cdots\}+G(z)\to f(x).$$

Similarly for $\phi(z)=z$, we obtain

$$G_1(z)\to f(x)=e^{ax}\{f(0)+f'(0)/a+f''(0)/a^2+\cdots\}\\ -\{f'(x)/a+f''(x)/a^2+f'''(x)/a^3+\cdots\}$$
$$=e^{ax}\{f(0)+f'(0)/a+f''(0)/a^2+\cdots\}+G(z)\to f(x);$$

and, more generally, for $\phi(z)=z^n$,

$$G_1(z)\to f(x)=a^{n-1}e^{ax}\{f(0)+f'(0)/a+f''(0)/a^2+\cdots\}\\ +G(z)\to f(x).$$

Assuming that $\phi(z)$ can be expressed in the form $\phi_0+\phi_1z+\phi_2z^2+\cdots$, we then derive by addition the result

$$G_1(z)\to f(x)=[\phi(a)/a]e^{ax}\{f(0)+f'(0)/a+f''(0)/a^2+\cdots\}\\ +G(z)\to f(x).$$

Since $U(x)=Ce^{ax}$ is obviously a solution of the equation $F(z)\to u(x)=0$, the truth of the theorem is demonstrated for the special case assumed above.

Let us next assume that $F(z)$ is of the form

$$F(z)=(z-a_1)(z-a_2)\cdots(z-a_n)/\phi(z),$$

where $a_1, a_2, \cdots, a_n$ are points within an annulus formed by two concentric

* The equivalence of the two expressions $e^{-ax}\to\{z^n\to f(x)\}$ and $z^ne^{-ax}\to f(x)$ is seen at once from Bourlet's formula (*loc. cit.*) if we let $G=e^{-ax}$ and $F=z^n$. Example: $z^{2m}e^{-ax}\to\sin x=\{z^{2m}-xz^{2m+1}+x^2z^{2m+2}/2!+\cdots\}\to\sin x=(-1)^m\{\sin x\cos x-\sin x\cos x\}=0$.

circles r and R, (the latter having the larger radius), about the origin and $\phi(z)$ has no singularity within or upon R.

Then the generatrix may be written

$$(3) \quad G(z) = \phi(z)\{1/P'(a_1)(z-a_1) + 1/P'(a_2)(z-a_2) + \cdots + 1/P'(a_n)(z-a_n)\},$$

where we employ the abbreviation

$$P(z) = (z-a_1)(z-a_2)\cdots(z-a_n).$$

Let us designate the expansion of the resolvant within the circle r by $G(z)$ and the expansion in the region exterior to R by $G_1(z)$. We then have from the result of the case of one pole, the expansion

$$(4) \quad G_1(z) \to f(x) = \sum_{i=1}^{n} [\phi(a_i)/a_i P'(a_i)] e^{a_i x}\{f(0) + f'(0)/a_i + f''(0)/a_i^2 + \cdots\} + G(z) \to f(x).$$

Hence the difference

$$U(x) = \{G_1(z) - G(z)\} \to f(x),$$

is a solution of the homogeneous equation.

Since we have

$d/dz\{P(z)/\phi(z)\}_{z=a_i} = F'(a_i)$, equation (4) can be written neatly as the contour integral

$$G_1(z) \to f(x) = (e^{-xz}/2\pi i)\int_C \{e^{xt}/(t-z)F(t)\}dt + G(z) \to f(x),$$

where C is a path around the zeros of $F(t)$.

The case of multiple poles is treated by a simple device. If the resolvant is

$$G(z) = \phi(z)/(z-a)^r,$$

we may write it in the form

$$G(z) = (\partial^{r-1}/\partial a^{r-1})\phi(z)/(z-a)(r-1)!.$$

Hence we have

$$G(z) \to f(x) = (\partial^{r-1}/\partial a^{r-1})[1/(r-1)!] \times \{[\phi(a)/a]e^{ax}[f(0) + f'(0)/a + f''(0)/a^2 + \cdots]\} + (-1)^r[\phi(z)/a^r]\{1 + rz/a + r(r+1)z^2/a^2 \cdot 2! + r(r+1)(r+2)z^3/a^3 \cdot 3! + \cdots\} \to f(x).$$

The difference between the left hand member and the second term of the right hand side is again seen to be a solution of the homogeneous equation.

It will be pointed out later that the development given above contains essentially a rationale of the Heaviside operational calculus which has played such an important rôle in the theory of electrical circuits.*

It remains for us to discuss the values of the solution and its derivatives at the point $x = 0$. We obtain the following theorem:

THEOREM 2. *If $G_1(z)$ denotes the Laurent expansion of the function*

$$G(z) = \phi(z)/(z-a_1)(z-a_2)\cdots(z-a_n),$$

in the region exterior to the poles $a_1, a_2, \cdots, a_n$, and if $\phi(z)$ is a polynomial of degree $m < n$, then $u_r(x) = z^r G_1(z) \to f(x)$ vanishes at $x = 0$ for $r = 0, 1, 2, \cdots, n-m-1$.

Proof: Writing $G(z)$ in the form (3) we have from the results of the last theorem,

$$u^{(r)}(x) = z^r G(z) \to f(x) = \sum_{i=1}^{n} [\phi(a_i) a_i{}^r e^{a_i x}/P'(a_i)]$$
$$\{f(0)/a_i + f'(0)/a_i{}^2 + \cdots\}$$
$$- \sum_{i=1}^{n} \phi(z)/P'(a_i) \to \{f^{(r)}(x)/a_i + f^{(r+1)}(x)/a_i{}^2 + \cdots\}.$$

Recalling the algebraic identity

$$I(p) = \sum_{i=1}^{n} a_i{}^p/P'(a_i) = \begin{cases} 1, & p = n-1, \\ 0, & 0 \leqq p < n-1, \end{cases}$$

we see that

$$u^{(r)}(0) = \sum_{i=1}^{n} \{\phi_0 + \phi_1 a_i + \cdots + \phi_m a_i{}^m\} a_i{}^r \{f(0)/a_i$$
$$+ f'(0)/a_i{}^2 + \cdots\}/P'(a_i) - \sum_{i=1}^{n} \sum_{j=0}^{m} \phi_j \{f^{(r+j)}(0)/a_i$$
$$+ f^{(r+j+1)}(0)/a_i{}^2 + \cdots\}/P'(a_i),$$
$$= \{\sum_{j=0}^{m} \phi_j I(r+j-1) f(0) + \sum_{j=0}^{m} \phi_j I(r+j-2) f'(0) + \cdots\}$$
$$- \{\sum_{j=0}^{m} I(-1)\phi_j f^{(r+j)}(0) + \sum_{j=0}^{m} I(-2)\phi_j f^{(r+j+1)}(0) + \cdots\},$$
$$= \sum_{j=0}^{m} \phi_j I(r+j-1) f(0) + \sum_{j=0}^{m} \phi_j I(r+j-2) f'(0) + \cdots$$
$$+ \sum_{j=0}^{m} \phi_j I(j) f^{(r-1)}(0).$$

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* See J. R. Carson, *Bulletin of the American Mathematical Society*, Vol. 31 (1926), pp. 43-68; also *Electric Circuit Theory and Operational Calculus*, McGraw-Hill, (1926).

If $r+j-1 \leqq n-2$, then $u^{(r)}(0)=0$. Since j does not exceed m we have $r \leqq n-m-1$, which is the statement of the theorem.

COROLLARY. *If $f(x)$ is a function which vanishes together with its first q derivatives at $x=0$, then $u^{(r)}(0)=0$ for $r=n+q-m$.*

3. *Application of Borel Summability.* We next seek conditions under which the function

$$u(x)=1/F(z) \to f(x)=G(z) \to f(x)$$

may exist and represent a solution of (1), where $G(z)$ is the Taylor expansion of the resolvant in a circle about the origin.

If a function $f(x)$ is unlimitedly differentiable we shall mean by its degree (stufe) the value L defined by the limit,

$$\lim_{n=\infty} L_n = \lim_{n=\infty} | f^n(x) |^{1/n} = L.$$

If L is bounded the following theorems can be readily deduced:

A. If $f(x)$ is of degree L, then $f^{(m)}(x)$ is of degree L.

B. If $f_1(x)$ and $f_2(x)$ are of degrees L_1 and L_2 respectively, where $L_2 > L_1$, then $a_1 f_1(x) + a_2 f_2(x)$ will be at most of degree L_2.

C. If $f(x)$ is of degree L, then $f(ax)$ is of degree $|a| L$.

D. If $f_1(x)$ is of degree L_1 and $f_2(x)$ is of degree L_2, then $f(x) = f_1(x) f_2(x)$ is at most of degree $L_1 + L_2$.*

A thorough treatment of the case where $f(x)$ is a function whose degree is finite may be found in the work of Schürer. It will be clear, however, that this restriction excludes from consideration a large class of equations of which the equation which defines the ψ function may be cited as typical:

$$u(x+1)-u(x)=1/x,$$

or in terms of a differential operator

$$(e^z-1) \to u = 1/x.$$

The following theorem will be found to extend the class of functions to which these operational methods may be applied:

THEOREM 3. *If in equation (1) $f(x)$ is of the form*

$$f(x)=g(x)+h(x),$$

where $g(x)$ is a function of finite degree L and where $h(x)$ is of the form

$$h(x)=h_1/x+h_2/x^2+h_3/x^3+\cdots,$$

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* See O. Perron, "Lineare Differentialgleichungen unendlich hoher Ordnung mit ganzen rationalen Koeffizienten," *Mathematische Annalen*, Vol. 84 (1921), pp. 31-32.

then a solution of (1) *exists of the form*

$$u(x) = \int_0^\infty e^{-xt} Q(t)/F(-t)\,dt + 1/F(z) \to g(x),$$

where

$$Q(t) = h_1 + h_2 t + h_3 t^2/2! + h_4 t^3/3! + \cdots, \tag{5}$$

provided, (a) $L < 1$, $L\rho < 1$, *where* ρ *is the radius of convergence of* $1/F(z)$; (b) *positive values* k, A, *and* M *exist such that* $|Q(t)/F(-t)| < Ae^{mt}$, *for* $0 < k \leqq t \leqq \infty$; *and* (c), $Q(t)/F(-t)$ *is of limited variation in the interval* $0 \leqq t \leqq k$.

It will be clear that the series $\sum_{i=1}^{\infty} b_i L^i$ forms a majorante for the series

$$U(x) = 1/F(z) \to g(x) = \{b_0 + b_1 z + b_2 z^2 + \cdots\} \to g(x),$$

since by hypothesis L is the degree of $g(x)$ and is smaller than the radius of convergence of the series expansion of $1/F(z)$. Since $U(x)$ is thus uniformly convergent we may form the derivative

$$U^{(n)}(x) = 1/F(z) \to z^n \to g(x). \tag{6}$$

From (A) above $z^n \to g(x)$ is of degree L and the series $L^n \sum_{i=1}^{\infty} b_i L^i$ forms a majorante for $U^{(n)}(x)$. It thus appears that $U(x)$ is at most of degree L.

To show that $F(z) \to U(x)$ converges uniformly to $g(x)$ let us write $S_n(x) = a_0 U(x) + a_1 U'(x) + \cdots + a_n U^{(n)}(x)$ and form the difference $A(x) = |g(x) - S_n(x)|$. Since $1/F(z) = b_0 + b_1 z + b_2 z^2 + \cdots$, we have $a_0 b_0 = 1$, $\sum_{i=0}^{k} a_i b_{r-i} \equiv 0$, $r > 0$. Hence we have the inequalities, $|\sum_{i=0}^{n} a_i b_{n+p-i}| < K\rho^p$ and $g^{(n)}(x) < K'L^n$, where K and K' are suitably chosen constants, and thus obtain

$$A(x) \leqq |\sum_{i=0}^{n} a_i b_{n+1-i}|\,|g^{(n+1)}(x)| + |\sum_{i=0}^{n} a_i b_{n+2-i}|\,|g^{(n+2)}(x)| + \cdots,$$

$$\leqq KK'L^{n+1}\rho/(1 - L\rho).$$

By hypothesis (a) we have $L\rho < 1$ and $L < 1$, and hence it follows that $A(x)$ converges uniformly to zero as $n \to \infty$.

We now consider the equation

$$V(x) = 1/F(z) \to h(x).$$

Applying the explicit expansion of the resolvant to $h(x)$ we get

$$V(x) = h_1\{b_0/x - b_1/x^2 + 2!\,b_2/x^3 - 3!\,b_3/x^4 + \cdots\}$$
$$+ h_2\{b_0/x^2 - 2!\;b_1/x^3 + 3!\,b_2/x^4 - 4!\,b_3/x^5 + \cdots\}$$
$$+ h_3\{b_0!/x^3 - 3!\,b_1/2!\,x^4 + 4!\,b_2/2!\,x^5 - 5!\,b_3/2!\,x^6 + \cdots\}$$
$$+ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

In general this series will be divergent, but it is usually summable by the method of Borel. This makes use of the identity $\int_0^\infty e^{-s}s^n ds = n!$ from which we obtain $V(x)$ in the form

$$V(x) = (1/x)\int_0^\infty [e^{-s}\{h_1 + h_2(s/x) + h_3(s/x)^2/2! + \cdots\}/F(-s/x)]\,ds.$$

Making the substitution $s = tx$, this becomes

$$V(x) = \int_0^\infty e^{-xt}Q(t)/F(-t)\,dt,$$

where $Q(t)$ is defined by (5).

The convergence of this result is easily established under hypotheses (b) and (c) stated in the theorem for we shall then have

$$|\,V(x)\,| < M/x + A\int_0^\infty e^{(m-x)t}dt = M/x + A/(x-m) \text{ for } x > m.$$

The case where $Q(t)/F(-t)$ has a pole of unit order in the interval $(0, \infty)$ is easily disposed of as follows:

Consider the function

$$G(t) = Q(t)/F(-t) - R/(a-t),$$

where R is the residue of $Q(t)/F(-t)$ at the point $t = a$. It is clear that $G(t)$ is regular at $t = a$ so we may consider the function

$$(7) \qquad V(x) = \int_0^\infty e^{-tx}G(t)\,dt + \int_0^\infty e^{-tx}R/(a-t)\,dt.$$

The second integral is divergent, but is seen to correspond formally for the case $m = 1$ to the solution of the equation,

$$(a - d/dx)^m \to W(x) = R/x,$$

which has for its particular integral the function

$$W(x) = (-1)^m R\int_k^x \{e^{a(x-t)}(x-t)^{m-1}/t(m-1)!\,\}dt.$$

More generally, if on the positive real axis $a_1, a_2, \cdots, a_m$ are poles of unit order of the function $Q(t)/F(-t)$, then the solution of the original equation will be

$$(8) \qquad V(x) = \int_0^\infty e^{-xt}G(t)\,dt \;- \sum_{i=1}^{m} R_i\int_{k_i}^x \{e^{a_i(x-t)}/t\}dt,$$

where the R_i are the residues of $Q(t)/F(-t)$ at the points a_i and

$$G(t) = Q(t)/F(-t) - \sum_{i=1}^{m} R_i/(a_i - t).$$

The case of poles of multiplicity m is treated in similar fashion by writing $G(t)$ in the form, $G(t) = Q(t)/F(-t) - \sum_{m=1}^{m} A_m/(a-t)^m$ where the A_m are the Laurent coefficients in the expansion of $Q(t)/F(-t)$. One then adds to the integral $\int_0^\infty e^{-tx}G(t)\,dt$ m functions obtained from $W(x)$ by letting m assume the values 1, 2, $3 \cdots m$ and replacing R by A_m.

Returning to the original problem, we see that the solution of the equation under the restrictions of the theorem is the sum, $u(x) = U(x) + V(x)$.

4. *Examples of the Theory.* Three examples will serve to illustrate the force of the theorems of the paper:

Example 1. Let us consider the equation

$$u(x+1) - u(x) = 1/x,$$

which may be written as $(e^z - 1) \to u(x) = 1/x$.

The resolvant may obviously be written as follows,

$$(9) \quad u(x) = 1/(e^z - 1) \to 1/x,$$
$$= \{1/z - 1/2 + B_1 z/2! - B_2 z^3/4! + B_3 z^5/6! - \cdots\} \to 1/x,$$

where $B_1 = 1/6$, $B_2 = 1/30$, $B_3 = 1/42, \cdots$ are the Bernoulli numbers.

From the well known relation $B_{p+1}/B_p > A(2p+1)(2p+2)$ where A is a constant,* it is clear that series (9) is divergent for all values of x.

But from (8) we see that the solution can be written in the form

$$(10) \qquad U(x) = c + \log x + \int_0^\infty e^{-tx}\{e^t/(1-e^t) + 1/t\}dt,$$

which, when $c = 0$, is a well known form for the function $\psi(x) = \Gamma'(x)/\Gamma(x)$.†

The expression given in (9) is readily seen to be the asymptotic expansion of (10). Making the transformation $t = s/x$ and integrating by parts we at once obtain

$$U(x) = c + \log x + (1/x)\{-1/2 + \sum_{p=1}^{n} (-1)^p B_p/2px^{2p-1} + (1/x^{2n+1}) \int_0^\infty e^{-t}\phi^{(2n+1)}(t/x)$$

* E. Borel, *Leçons sur les séries divergentes*, Paris (1901), p. 24.

† See N. Nielsen, "Handbuch der Theorie der Gammafunktionen," Leipzig (1906), p. 183.

where $\phi(v) = e^v/(1-e^v) + 1/v$.

Consider the remainder

$$R_n = |\ (1/x^{2n+1}) \int_0^\infty e^{-t}\phi^{(2n+1)}(t/x)\,dt\ |.$$

Since we have

$$\phi^{(n)}(v) = \sum_{r=1}^{\infty} A_r e^{rv}/(1-e^{rv})^r + (-1)^n(n-1)!/v^n,$$

where A_r are functions of n alone it is clear that for large values of v, $\phi^{(n)}(v) \sim \sum_{r=1}^{n} (-1)^r A_r$. From this it follows that positive values A and m exist such that we have $|\ \phi^{(2n+1)}(v)\ | < Ae^{mv}$. Hence for $x > m$, we get

$$R_n < (1/x^{2n+1}) \int_0^\infty e^{-t} A\ e^{mt/x} dt = A/x^{2n}(x-m).$$

Consequently it follows that

$$\lim_{|x|=\infty} x^{2n} R_n < \lim_{|x|=\infty} A/(x-m) = 0,$$

which satisfies Poincaré's criterion for the asymptotic expansion of $U(x)$.*

Example 2. The fundamental equation satisfied by the polynomials representing the sums of powers of the natural numbers, $S_n(p) = 1^n + 2^n + \cdots + p^n$, is obviously †

$$S_n(p) - S_n(p-1) = p^n,$$

which may be written symbolically as

$$(1-e^{-z}) \to S_n(p) = p^n.$$

The solution follows immediately:

$$\begin{aligned} S_n(p) &= 1/(1-e^{-z}) \to p^n, \\ &= \{1/z + 1/2 + B_1 z/2! - B_2 z^3/4! + B_3 z^5/6! - \cdots\} \to p^n, \\ &= p^{n+1}/(n+1) + p^n/2 + nB_1 p^{n-1}/2! - n(n-1)(n-2)B_2 p^{n-3}/4! \\ &\quad + n(n-1)(n-2)(n-3)(n-4)(n-5)B_3 p^{n-5}/6! - \cdots, \end{aligned}$$

which is immediately recognized as Bernoulli's famous formula.

Example 3. The Heaviside expansion theorem so useful in the study of electrical circuits is an immediate consequence of equation (4).‡

The problem studied by Heaviside was that of determining the current in an electrical net work when an electromotive force $E(t)$ is impressed on the net work at time $t = 0$. The special feature of this problem is that the unknown function (or functions in the case of multiple net works) denoting

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* *Acta Mathematica*, Vol. 8 (1886), pp. 295-344.

† See Nielsen, *Traité des nombres de Bernoulli*, Paris (1923), pp. 296-297.

‡ For details and the extensive literature on this theorem see Carson, *loc. cit.*

the current in the net work must vanish together with its derivatives to as high an order as possible at $t = 0$. From theorem 2 it is clear that this property is possessed by the solution (4) which, consequently, should be identifiable with the Heaviside expansion theorem.

That this is the case is immediately seen if we change variables from x to t and let $f(t) = E(t) = 1$. We then get

$$u(t) = G(z) \to E(t) = \sum_{k=1}^{n} e^{a_k t}/a_k F'(a_k) + 1/F(0),$$

which is the Heaviside theorem.

Carson's extension of the Heaviside theorem to an alternating E. M. F. is at once obtained if we let $E(t) = e^{int}$. We thus have

$$u(t) = G(z) \to E(t) = \sum_{k=1}^{n} e^{a_k t}/(a_k - in)F'(a_k) + e^{int}\,1/F(in).$$

The form of solution of the general problem found most useful by Carson is derived in a simple manner from equation (4).

Making use of the fact that in the development

$$1/F(z) = b_0 + b_1 z + b_2 z^2 + \cdots,$$

the coefficients may be calculated from the formulas

$$b_0 = 1/F(0), \qquad b_m = -\sum_{k=1}^{n} 1/a_k^{m+1} F'(a_k), \qquad m > 0,$$

where the a_k are the roots of $F(z) = 0$, we may write

$$\begin{aligned} u(t) = \{ &\sum_{k=1}^{n} e^{a_k t} E(0)/a_k F'(a_k) + \sum_{k=1}^{n} e^{a_k t} E'(0)/a_k^2 F'(a_k) \\ &+ \sum_{k=1}^{n} e^{a_k t} E''(0)/a_k^3 F'(a_k) + \cdots\} + E(t)/F(0) \\ &- \sum_{k=1}^{n} E'(t)/a_k^2 F'(a_k) - \sum_{k=1}^{n} E''(t)/a_k^3 F'(a_k) - \cdots, \\ = &\sum_{j=1}^{\infty} \sum_{k=1}^{m} \{e^{a_k t} E^{(j-1)}(0)/a_k^j F'(a_k) - E^{(j-1)}(t)/a_k^j F'(a_k)\} \\ &+ E(t)/F(0) + \sum_{k=1}^{n} E(t)/a_k F'(a_k), \\ = &\int_0^t \{\sum_{k=1}^{n} e^{a_k (t-s)} E(s)/F'(a_k)\} ds + E(t)/F(0) + \sum_{k=1}^{n} E(t)/a_k F'(a_k). \end{aligned}$$

This is easily seen to be equivalent to the expansion

$$u(t) = z \to \int_0^t \{\sum_{k=1}^{n} e^{a_k (t-s)}/a_k F'(a_k) + 1/F(0)\} E(s)\, ds,$$

which is Carson's formula.

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On Generalized Tchebycheff Inequalities in Mathematical Statistics.

By Clarence DeWitt Smith.*

1. *Introduction.* If P is the probability that a variable will deviate from its expected value by an amount as great as t times the standard deviation of the variable, the Tchebycheff inequality states that P is not greater than $1/t^2$ for every t.† This inequality may be regarded as a criterion which places an upper bound on the probability of a deviation of an item from its expected value without regard to the nature of the distribution from which the item is drawn. Bienaymé contributed to the development of such a criterion in a paper ‡ published in 1853. Several contributions have been made in recent years to the generalization of the inequality with a view to obtaining closer inequalities both in the general case where no restrictions are placed on the set of positive values and in cases where certain restrictions are placed on the distribution of the given values. The first important step in generalizing the inequality appeared in a paper § by Karl Pearson in 1919. He obtains the inequality

$$P_{t\sigma} > 1 - M_{2n}/t^{2n}M_2^n$$

where M_{2_n} is the $2n^{th}$ moment about the mean and $P_{t\sigma}$ is the probability that a deviation from the mean of the distribution is less than t times the standard deviation if t is a positive number.

The inequality is further generalized in a paper ¶ by B. H. Camp in 1922. By placing a mild restriction on the nature of the frequency function he succeeded in reducing the larger member of the Pearson inequality by about fifty per cent. The main restriction placed on $f(x)$ is that it be a monotonic

* Acknowledgments are due to Professor H. L. Rietz for his helpful suggestions in regard to certain important points discussed in the paper.

† Tchebycheff, "Des Valeurs Moyennes," *Journal de Mathématiques* (2), Vol. 12 (1867), pp. 177-84.

‡ M. Bienaymé, "Considerations à l'appue de la découverte de Laplace sur la loi de probabilité dans la méthode des moindres carrés," *Comptes Rendus*, Vol. 37 (1853), pp. 309-24.

§ Karl Pearson, "On Generalized Tchebycheff Theorems in the Mathematical Theory of Statistics," *Biometrika*, Vol. 12 (1918-19), pp. 284-96.

¶ B. H. Camp, "A New Generalization of Tchebycheff's Statistical Inequality," *Bulletin of the American Mathematical Society*, Vol. 28 (1922), pp. 427-32.

decreasing function of $|x|$ when $|x| \geqq c\sigma$, $c \geqq 0$. With the origin chosen so that zero is at the mean of the distribution he obtains the generalized inequality

$$P_{t\sigma} \leqq \frac{\beta_{2n-2}}{t^{2n}(1+1/2n)^{2n}} \cdot \frac{1}{1+\phi} + \theta.$$

In this inequality $P_{t\sigma}$ is the probability that $|x|$ is as great as $t\sigma$ when t is a positive number,

$$\theta = \frac{\phi \cdot P_{c\sigma}}{1+\phi} \quad \beta_{2n-2} = \frac{M_{2n}}{\sigma^{2n}} \text{ and } \phi = \frac{[2cn/t(2n+1)]^{2n}}{(2n+1)(t/c-1)}.$$

When P_d is the probability that a deviation from the most probable value is as great as $d = tM_n^{1/n}$ the above inequality for $c = 0$ is

$$P_d \leqq 1/t^n(1+1/n)^n.$$

This is the inequality due to M. B. Meidell.* Results analogous to those of Pearson and Meidell were published in 1923 in a paper † by Narumi. He represents by $f(x)$ any law of frequency of infinite range and assumes a set of positive deviations from an arbitrary origin by setting

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^{\infty} [f(x) + f(-x)]\,dx = 1.$$

He also arrives at certain closer inequalities for an increasing function.

It is the main purpose of the present paper to give a further development of the theory and properties of what may be appropriately called generalized Tchebycheff inequalities. We shall give in Section 2 a simple development of the generalized inequality with no restrictions on the nature of the distribution of a set of positive values. The inequality regarded as a proposition of geometry will be discussed in Section 3. In Section 4 we shall derive some closer inequalities, give a discussion of their properties, and a discussion of the effects on the inequality of certain restrictions on the nature of the distribution function. Special functions for which the inequality may be made as close as you please will be discussed in Section 5. The fact that the closeness of the inequality may be improved in certain cases by moving the origin

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* M. B. Meidell, "Sur un problème du calcul des probabilités et les statistiques mathématiques," *Comptes Rendus*, Vol. 175 (1922), p. 806; also, "Sur la probabilité des erreurs," *Comptes Rendus*, Vol. 176 (1923), p. 280; see also *Skandinavisk Aktuarietidskrift*, årgång V (1922), p. 210.

† Seimatsu Narumi, "On Further Inequalities with Possible Application to Problems in the Theory of Probability," *Biometrika*, Vol. 15 (1923), p. 245. See also M. Alf Guldberg, "Sur le théorème de M. Tchebycheff," *Comptes Rendus*, Vol. 175 (1922), p. 418.

will be discussed in Section 6. In Section 7 we shall discuss certain types of distributions for which the difference between the members of the inequality attains a minimum.

2. *Development of the Inequality from Markoff's Lemma.* The inequality has been developed by different authors from somewhat different viewpoints. Tchebycheff assumed a set of deviations of variables from their expected values. Pearson and Camp assumed a set of deviations from the arithmetic mean and considered moments of even order. Meidell assumed a set of deviations from the most probable value. Narumi assumed a set of positive deviations from an arbitrary origin. Hence it should be of interest at this point to develop a generalized inequality which gives rise to several important known forms of the generalized Tchebycheff inequality in a very simple way. We shall begin by giving a proof of a generalized form of the Markoff Lemma * which may be stated in the following modified form.

If a variable U takes any set of positive values, $u_1, u_2, \cdots, u_k$ with the corresponding probabilities, $p_1, p_2, \cdots, p_k$ respectively, and if M_n is the expected value of U^n, then the probability P_d that U is greater than d is not greater than M_n/d^n where d is a positive number.

To prove this generalized form of the lemma let $u_c < d \leqq u_{c+1}$. Then the probability that U is greater than d is $P_d = p_{c+1} + p_{c+2} + \cdots + p_k$.

$$M_n = U_1{}^n p_1 + U_2{}^n p_2 + \cdots + U_c{}^n p_c + U^n{}_{c+1} p_{c+1} + \cdots + u_k{}^n p_k$$
$$\geqq d^n (p_{c+1} + p_{c+2} + \cdots + p_k) = d^n P_d.$$

Hence

$$P_d \leqq M_n/d^n. \tag{1}$$

Now let us consider the forms of the inequality which result from certain assigned values of n and d. First let $n = 2$ and $d = tM_2{}^{1/2}$ where M_2 is the second moment of deviations of a variate from its expected value. Then,

$$P_d \leqq 1/t^2, \tag{2}$$

which is the Tchebycheff inequality for a single variate. Next let n be an even number and $d = tM_2{}^{1/2}$ where M_2 is the second moment of deviations of a variate from the arithmetic mean. Then $M_2{}^{1/2}$ is the standard deviation σ, and

$$P_d \leqq M_n/t^n (M_2{}^{1/2})^n \tag{3}$$

gives a bound analogous to that of Pearson. Finally, if $d = tM_n{}^{1/n}$,

$$P_d \leqq 1/t^n. \tag{4}$$

* A. A. Markoff, *Wahrscheinlichkeitsrechnung* (1912), p. 54.

This inequality is analogous to that of Narumi. Hence we have shown how the generalized inequality (1) gives rise to several of the important known inequalities in a rather simple way.

3. *The inequality regarded as a proposition of geometry.* The theorem of Tchebycheff and its generalizations may be regarded as criteria which hold between the area and the moments of area under a curve. If $y = f(x)$ is a single valued continuous function of x, then for a certain area in the first quadrant the theorem may be stated as an inequality which holds between the moments of area under the curve and the area under a given part of the curve. When so stated it may be put in the following form.

If we take a unit of area in the first quadrant bounded by a curve $y = f(x)$, the x-axis, and the lines $x = a$ and $x = k$, $(k > a)$, then the nth moment M_n, $(n > 0)$, of this unit area about the y-axis is not less than the product of d^n and the part of this area to the right of the line

$$x = d, \quad (M_n^{1/n} < d \leqq k).$$

4. *On some further relatively close inequalities.* It is shown in Section 7 for any distribution of infinite range that the two members of the Tchebycheff inequality will approach arbitrarily near to equality as d is increased indefinitely. But the inequality is not in general close to an equality when d is a finite number of the order of $M_n^{1/n}$ except for specially designed functions. It would seem to be of interest to obtain a closer inequality connecting d, P_d, and the nth power of the values of the variable applicable to a wide class of functions, and if possible to obtain an inequality that will in general approach an equality at the ends of a bounded distribution.

Consider again the increasing sequence of positive numbers $u_1, u_2, u_3, \cdots, u_k$ which are the values taken by a variable U with the respective probabilities $p_1, p_2, p_3, \cdots, p_k$. Following the notation in Section 2, let d be chosen so that $u_c < d \leqq u_{c+1}$ and let $M_n = \sum_{i=1}^{k} u_i^n p_i$ be the expected value of U^n. Then it follows that

$$\sum_{i=c+1}^{k} u_i^n p_i \geqq d^n \sum_{i=c+1}^{k} p_i, \qquad (n > 0).$$

But $\sum_{i=c+1}^{k} p_i = P_d$, the probability that a value of U taken at random is greater than d. Hence,

$$P_d \leqq (1/d^n) \sum_{i=c+1}^{k} u_i^n p_i. \tag{5}$$

If $f(x) \geqq 0$ is a probability function that is continuous and integrable from d to k, this inequality takes the form

$$P_d \leqq (1/d^n) \int_d^k x^n f(x)\,dx, \tag{6}$$

where

$$P_d = \int_d^k f(x)\,dx.$$

Next let $x_m = \sum_{i=c+1}^{k} u_i p_i / \sum_{i=c+1}^{k} p_i$ be the mean value of $u_{c+1}, u_{c+2}, \cdots, u_k$, weighted with their respective probabilities. Since it is well known * that for $n \geqq 1$,

$$\sum_{i=c+1}^{k} u_i^n p_i / \sum_{i=c+1}^{k} p_i \geqq (\sum_{i=c+1}^{k} u_i p_i / \sum_{i=c+1}^{k} p_i)^n = x_m^n,$$

we have

$$P_d \leqq (1/x_m^n) \sum_{i=c+1}^{k} u_i^n p_i. \tag{7}$$

Similarly for the continuous probability function $f(x)$ we have

$$P_d \leqq (1/x_m^n) \int_d^k x^n f(x)\,dx. \tag{8}$$

It seems desirable to emphasize that the inequality (7) is in general a closer inequality than (5) because $x_m > d$ for the increasing sequence. When $f(x)$ is an increasing function throughout the interval d to k, x_m is greater than $\frac{1}{2}(k+d)$. Therefore for a function $f(x)$ that increases throughout the interval d to k it follows that

$$P_d \leqq [2/(d+k)]^n \int_d^k x^n f(x)\,dx. \tag{9}$$

In the inequalities (7), (8), and (9) it should be noted that $n \geqq 1$. In all cases where the function $f(x)$ increases from d to k, (9) holds and is a closer inequality than (6). To compare the relative values of the actual probability and the values given by (6) and (9) we may take as a simple case the function $f(x) = x^2$. Let $2\int_0^k x^2 dx = 1$, $n = 4$, and we have

d	P	(6)	(9)
1.1	.112	.1223	.1135
1.5	0	0	0

A comparison of these values shows the very close approximation to P as given by (6) and (9). In applying the inequality (9) it is well to note that there is a limitation on the value of k in terms of n and M_n in the case

* Chrystal's *Algebra*, Part II, Par. 9, p. 48.

of a non-decreasing frequency function. For if we assume a unit area bounded by $y = f(x)$, the x-axis, the y-axis, and the line $x = k$ where $f(x)$ is a non-decreasing function in the interval $x = 0$ to $x = k$ then *

$$k \leqq (1+n)^{1/n} M_n^{1/n}; \qquad n \geqq 1. \tag{10}$$

To establish this inequality let

$$\bar{y} = (1/k) \int_0^k y dx = 1/k$$

be the mean ordinate of the curve $y = f(x)$. For the nth moment of the rectangle $OBCR$, where $OB = 1/k$, we have

$$M'_n = (1/k) \int_0^k x^n dx = k^n/(n+1).$$

Hence for the constant distribution we have

$$k = (1+n)^{1/n} (M'_n)^{1/n}.$$

Since $y = f(x)$ is a non-decreasing function in the interval $0 \leqq x \leqq k$ it is geometrically fairly obvious for $n \geqq 1$, that the nth moment of the area under the curve $y = f(x)$ is not less than the nth moment of the rectangle. That is, $M_n^{1/n} \geqq (M'_n)^{1/n} = k(1+n)^{-1/n}$, or $k \leqq (1+n)^{1/n} M_n^{1/n}$. It now follows that d for non-decreasing functions is restricted in accord with the inequality, $M_n^{1/n} < d \leqq k \leqq (1+n)^{1/n} M_n^{1/n}$.

While the right member of the inequality (5) sets an upper bound for the probability P_d that seems ordinarily more difficult to compute than $P_d = \sum_{i=c+1}^{k} p_i$ itself, nevertheless (5) serves as a starting point for the development of certain useful inequalities. By sacrificing something as regards the smallness of the upper bound in the inequality (5) we may write the inequality in the form

$$P_d \leqq (1/d^n)[\sum_{i=1}^{k} u_i^n p_i - \psi_d], \tag{11}$$

where

$$\psi_d \leqq \sum_{i=1}^{c} u_i^n p_i \quad (u_c < d \leqq u_{c+1}).$$

For a simple case, we may take

$$\psi_d = u_1^n \sum_{i=1}^{c} p_i = u_1^n (1 - P_d),$$

since $u_1 < u_2 < \cdots < u_c$.

* Narumi, *loc. cit.*, p. 249.

Substitution of this value in (11) gives

$$P_d \leqq (\sum_{i=1}^{k} u_i^n p_i - u_1^n)/(d^n - u_1^n). \tag{12}$$

Similarly, for the case of a continuous function $y = f(x)$ with a range a to k $(a < k)$, we write in place of (11),

$$P_d \leqq (1/d^n)[\int_a^k x^n f(x)\,dx - \psi_d], \tag{13}$$

where

$$\psi_d \leqq \int_a^d x^n f(x)\,dx.$$

By taking $\psi_d = a^n \int_a^d f(x)\,dx = a^n(1 - P_d)$ we have

$$P_d \leqq (M_n - a_n)/(d^n - a^n). \tag{14}$$

In other cases we may take $\psi_d = \int_a^d x^n y\,dx$, where $y = f_1(x)$ is a function of convenient form and such that

$$\int_a^d x^n f_1(x)\,dx \leqq \int_a^d x^n f(x)\,dx,$$

For example, it may lead to useful results in certain cases to take $y = f_1(x)$ as a suitably selected tangent or chord of the curve $y = f(x)$, the selection to be made depending on the nature of the function $f(x)$ in the interval $x = a$ to $x = k$.

Although such a direct treatment of the problem is possible we have found it convenient to proceed with a less direct plan for obtaining a closer inequality than $P_d \leqq (M_n/d^n)$ by first changing somewhat the form of the inequality (6). For the present we shall limit our considerations to the continuous function $f(x) = 0$ in the interval $0 \leqq x \leqq k$ such that $\int_0^k f(x)\,dx = 1$. While we take x positive, it is fairly obvious that any frequency curve $y = g(x)$ in which x takes negative as well as positive values may be included by taking $f(x) = g(x) + g(-x)$. By using moments $M_{2r} = \int_0^k x^{2r} f(x)\,dx$ of even order $2r$ we obtain from (6),

$$P_d \leqq [M_{2r} - \int_0^d x^{2r} f(x)\,dx]/d^{2r} = 1/t^{2r} - R, \tag{15}$$

where $r \geqq 1$, $d = t(M_{2r})^{1/2r}$, and $R = (1/d^{2r}) \int_0^d x^{2r} f(x)\,dx$. Now we may derive further inequalities by substituting for R any smaller number. We shall attempt to make such substitutions as will lead to interesting results.

Let $\phi(x)$ be such that $(d/dx)[\phi(x)] = f(x)$ and let $P_x = \int_x^k f(x)dx$. Then $\phi(x) = -P_x$, $\int_0^d x^{2r}f(x)dx = -d^{2r}P_d + 2r\int_0^d x^{2r-1}P_x dx$, and

$$R = -P_d + (2r/d^{2r})\int_0^d x^{2r-1}P_x dx. \tag{16}$$

From (15) and (16),

$$2r\int_0^d x^{2r-1}P_x dx \leqq M_{2r}. \tag{17}$$

Since $(d/dx)P_x = -f(x)$ is negative the second derivative of P_x is positive when $f(x)$ is a decreasing function. Let us consider the case where $f(x)$ is a decreasing function in the interval $0 \leqq x \leqq d$. In this case the curve $y = P_x$ is concave upward. The equation of a tangent to the curve $y = P_x$ at $x = \theta d$, $(0 \leqq \theta \leqq 1)$ is

$$y = -xf(\theta d) + P_{\theta d} + \theta d f(\theta d). \tag{18}$$

Since $y = P_x$ is concave upward it does not cross this tangent and the inequality (17) remains valid if we replace P_x by the right member of (18). This gives after integration,

$$-[2r/(2r+1)]\, d^{2r+1}f(\theta d) + d^{2r}[P_{\theta d} + \theta d f(\theta d)] \leqq M_{2r},$$

or

$$P_{\theta d} + df(\theta d)[\theta - 2r/(2r+1)] \leqq M_{2r}/d^{2r}. \tag{19}$$

The closeness of this inequality depends on the selection of θ. To find a value of θ which would make the non-negative difference between the members of (19) a minimum we write

$$V = M_{2r}/d^{2r} - P_{\theta d} - df(\theta d)[\theta - 2r/(2r+1)],$$

and seek the value of θ which makes

$$dV/d\theta = df(\theta d) - df(\theta d) - d^2 f'(\theta d)\,[\theta - 2r/(2r+1)] = 0.$$

This equation is satisfied by $\theta = 2r/(2r+1)$. The second derivative of V is positive for this value of θ and the condition for a minimum is satisfied. If we let $\theta = 2r/(2r+1)$ in (19),

$$P_{2rd/2r+1} \leqq M_{2r}/d^{2r}.$$

A simple transformation gives

$$P_d \leqq M_{2r}/d^{2r}(1 + 1/2r)^{2r}.$$

If $d = t(M_{2r})^{1/2r}$,

$$P_d \leqq 1/t^{2r}(1 + 1/2r)^{2r}. \tag{20}$$

This is the inequality of Meidell and of B. H. Camp for his $c = 0$. We have proved that this inequality is closer to an equality than any other derivable by using in place of P_x in (17) the function represented by a tangent to the curve $y = P_x$ at any point of the interval $x = 0$ to $x = d$.

Next let us consider the case in which $f(x)$ is an increasing function. In this case we know from (10) that the range 0 to k for consideration is limited to $t \leqq (1 + 2r)^{\frac{1}{2}r}$. Now the second derivative of P_x is negative and the curve $y = P_x$ is concave downward. Consider the function represented by a chord which connects the points on $y = P_x$ for which $x = 0$ and $x = d$. The equation of this chord is

$$y = (x/d)(P_d - 1) + 1. \tag{21}$$

When we substitute the right member of (21) for P_x in (17),

$$(2r/d)(P_d - 1)\int_0^d x^{2r}dx + 2r\int_0^d x^{2r-1}dx \leqq M_{2r}.$$

After integration this equation reduces to

$$P_d \leqq (M_{2r}/d^{2r})(1 + 1/2r) - 1/2r.$$

When $d = t(M_{2r})^{1/2r}$,

$$P_d \leqq (1/t^{2r})(1 + 1/2r) - 1/2r. \tag{22}$$

On comparing the upper bound in (22) with the upper bound set by the inequality of Narumi * for an increasing function, which is

$$[1 - b^n/(n+1)]/[t^n - b^n/(n+1)];\ t \geqq b,$$

it seems that they are equal when $t = b$ and that the upper bound in (22) is smaller when $t > b$.

Let us consider next the case in which $y = f(x)$ is an increasing function when $0 < x < h$ $(0 < h < k)$, and a decreasing function when $x \geqq h$. Then the function $Y = P_x$ will be represented by a curve which is concave downward from $x = 0$ to $x = h$ and concave upward beyond $x = h$. We may substitute for $\int_0^d x^{2r-1}P_x dx$ in (17) the moment under the horizontal line AB from zero to h plus the moment under the tangent at $x = \theta d$ from h to d where as above $0 \leqq \theta \leqq 1$. We will then have from (17)

$$2r\int_0^h x^{2r-1}P_h dx + 2r\int_h^d x^{2r-1}[-xf(\theta d) + P_{\theta d} + \theta d f(\theta d)]dx$$
$$= P_h h^{2r} + \{-[2rd^{2r+1}/(2r+1)]f(\theta d) + d^{2r}[P_{\theta d} + \theta d f(\theta d)]$$
$$+ [2rh^{2r+1}/(2r+1)]f(\theta d) - h^{2r}[P_{\theta d} + \theta d f(\theta d)]\} \leqq M_{2r},$$

* *Loc. cit.*, p. 246.

or

$$(23)\quad P_{\theta d} + \theta d f(\theta d) - [2r/(2r+1)]\,[(d^{2r+1} - h^{2r+1})/(d^{2r} - h^{2r})]f(\theta d) \leqq (M_{2r} - h^{2r}P_h)/(d^{2r} - h^{2r}).$$

We may ask for the value of θ which will make the non-negative difference between the two members of (23) a minimum. Let the difference be represented by

$$V = (M_{2r} - h^{2r}P_h)/(d^{2r} - h^{2r}) - P_{\theta d} - \{\theta d - [2r/(2r+1)]\,[(d^{2r+1} - h^{2r+1})/(d^{2r} - h^{2r})]\}f(\theta d).$$

Then if we set $dV/d\theta = 0$, we find that the value of θ which satisfies the condition for a minimum is

$$(24)\qquad d' = \theta d = [2r/(2r+1)]\,[(d^{2r+1} - h^{2r+1})/(d^{2r} - h^{2r})].$$

By substituting in (23) the value of θd given by (24),

$$P_{d'} \leqq (M_{2r} - h^{2r}P_h)/[(d'/\theta)^{2r} - h^{2r}].$$

We may let $h = c(M_2)^{\frac{1}{2}}$, and if $y = f(x)$ is the distribution of absolute values from an arithmetic mean, we have $M_2^{\frac{1}{2}} = \sigma$, where σ is the standard deviation. Then if $d' = t\sigma$ we have

$$(25)\qquad P_{d'} \leqq (\beta_{2r-2} - c^{2r}P_{c\sigma})/[(t/\theta)^{2r} - c^{2r}],$$

where

$$\beta_{2r-2} = (M_{2r}/\sigma^{2r}).$$

Here θ is determined from the equation

$$t = [2r/(2r+1)]\,[t^{2r+1} - (c\theta)^{2r+1}]/\theta[t^{2r} - (c\theta)^{2r}].$$

Now we know that $P_{c\sigma}$ is greater than $P_{t\sigma}$ and if we substitute $P_{t\sigma}$ for $P_{c\sigma}$ in (25),

$$(26)\qquad P_{t\sigma} \leqq \beta_{2r-2}/(t/\theta)^{2r}.$$

If we set $c = r = 1$ and $t = 2$ in (26) we get $P_{2\sigma} \leqq .133$.

Referring again to the function $y = P_x$ with the assumptions which lead to (23) we find that the equation of the chord $A'B$ from $x = 0$ to $x = h$ is

$$y = (1/h)[x(P_h - 1) + h].$$

If we integrate from $x = 0$ to $x = h$ using this function, the first integral in the development which lead to (23) will be

$$2r\int_0^h (1/h)[x(P_h - 1) + h]\,x^{2r-1}dx = h^{2r}(2rP_h + 1)/(2r+1).$$

Proceeding as above we obtain in place of (25),

$$(27) \qquad P_{t\sigma} \leqq \{\beta_{2r-2} - c^{2r}[(2rP_{c\sigma} + 1)/(2r+1)]\}/[(t/\theta)^{2r} - c^{2r}].$$

We may again substitute $P_{t\sigma}$ for $P_{c\sigma}$ as in the preceding case and

$$(28) \qquad P_{t\sigma} \leqq [\beta_{2r-2} - c^{2r}/(2r+1)]/[(t/\theta)^{2r} - c^{2r}/(2r+1)],$$

where θ is determined as in (25). If $c = r = 1$ and $t = 2$ in (28) we have $P_{2\sigma} \leqq .092$. The inequality of Camp for the same values gives $P_{2\sigma} \leqq .143$.

5. *On close inequalities for special functions.* Narumi * states that functions exist for which the difference between the two members of Pearson's inequality may become arbitrarily small. In a recent paper † Bortkiewicz presents a discussion of the Markoff Lemma in which he gives a method of determining a distribution for which the difference between the members of the inequality stated in the lemma may be made equal to a previously assigned small number. A similar method may be applied to a generalized Tchebycheff inequality. In the present section we shall present a discussion of a method applicable to the generalized Tchebycheff inequality

$$P_d \leqq M_n/d^n,$$

in such a way as to bring out the point that the equality is not realized except as a limiting case.

Assume a distribution of positive varieties composed of the parts a and $1-a$. Let the variates which belong to $1-a$ be distributed in any convenient manner from the origin to a point at a distance d_1 from the origin. Then place the variates which belong to a, at a distance $d > d_1$. If M_n' is the nth moment of that part of the distribution represented by $1-a$,

$$M_n = M_n' + ad^n, \text{ or } M_n/d^n = (M_n'/d^n) + a.$$

It is now fairly obvious that as d_1 is decreased we can make M_n' approach zero and M_n/d^n will approach a. But $P_{d-\epsilon} = a$ for $\epsilon < d - d_1$ and $M_n/(d-\epsilon)^n$ approaches M_n/d^n as ϵ becomes arbitrarily small. Hence the difference, $M_n/(d-\epsilon)^n - P_{d-\epsilon}$, becomes arbitrarily small for sufficiently small values of ϵ.

The following distributions have been found to give rather close values for the inequality, and they seem to suggest probable forms of a continuous

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* *Loc. cit.*, p. 245.

† L. V. Bortkiewicz, "Zum Markoffschen Lemma," *Skandinavisk Actuarietidskrift*, Häft 1 (1927), p. 13.

function which would have a point at which the inequality may be expected to become fairly close.

p_i	.488	.005	.004	.0025	.0005	0	.0005	.0025	.004	.005	.488
x_i	0	1	2	3	4	5	6	7	8	9	10

Let $tM_2^{\frac{1}{2}} = 10 - \epsilon$. Then $P = .488$ and $1/t^2 \leqq .4965$.

p_i	0	.0005	.0006	.0007	.0008	.0009	.9965
x_i	0	1	2	3	4	5	6

Let $tM_2^{\frac{1}{2}} = 6 - \epsilon$. Then $P = .9965$ and $1/t^2 \leqq .9977$.

In these illustrations ϵ is a positive number small at pleasure.

6. *On certain changes in the Tchebycheff inequality produced by moving the origin.* It is well known that the Tchebycheff inequality and certain of its generalizations are valid for positive deviations from any origin. However the degree of approach toward equality of the two members of the inequality may depend upon the origin from which deviations are measured. If $|x_1|, |x_2|, \cdots |x_k|$, are non-negative deviations from any origin it is well known that

$$P_d \leqq M_n/d^n, \tag{29}$$

where P_d is the probability that an $|x|$ taken at random from the population is as great as d.

In considering the degree to which the right hand member of (29) can be made to approach the left hand member by a suitable selection of an origin from which to measure deviations we shall assume that the original origin is such that we have a set of non-negative deviations. We propose first to consider the question of moving the origin so as to change the value of each deviation by an amount h where h may take positive values and negative values such that $|h|$ is not greater than the smallest deviation from the original origin. Now if P_d is the probability of a deviation as great as d from the original origin, $P_d = P'_{d+h}$ where P'_{d+h} is the probability of a deviation as great as $d + h$ from the new origin. Then in order to find the origin for which the right hand member of the inequality

$$P'_{d+h} \leqq \sum_{i=a}^{k} (|x_i| + h)^n/(d + h)^n$$

shall approach as near as possible to the left hand member we seek the value of h which will make

$$V_1 = \Sigma (|x_i| + h)^n/(d + h)^n \tag{30}$$

a minimum. Such a value of h will satisfy the equation

$$(31) \qquad dV_1/dh = \frac{n(d+h)[\Sigma(|x_i|+h)^{n-1}] - n\Sigma(|x_i|+h)^n}{(d+h)^{n+1}} = 0.$$

If $n=2$ in (31),

$$(32) \qquad h = (\Sigma|x_i|^2 - d\Sigma|x_i|)/(Nd - \Sigma|x_i|).$$

If the original set of values were the absolute values of deviations from an arithmetic mean we may well write (32) in the form

$$(33) \qquad h = (\sigma^2 - dD)/(d - D),$$

where σ is the standard deviation and D is the mean deviation from the arithmetic mean. An examination of (33) shows that h is ordinarily positive for $d < \sigma^2/D$ and negative for $d > \sigma^2/D$; for, in general, $d - D$ is positive since, in general, $d > \sigma > D$.

To illustrate by a simple numerical case, let us take x as the absolute values of deviations from an arithmetic mean with the following distribution:

Frequency	2,	6,	d,	2,
$\lvert x_i \rvert$	0,	1,	1.3,	2.

Here $\sigma^2 = 1.4$, $d = 1.3$, $D = 1$, and $h = 1/3$. Next consider the distribution

Frequency	0	2	6	d	2,
$\lvert x_i \rvert$	0	1	2	2.3	3.

In this case $\sigma^2 = 4.4$, $d = 2.3$, $D = 2$, and $h = -2/3$, $P_d = .832$, and $P'_{d+h} = .812$. These results show the amount of improvement in the value of the approximation in this very simple case.

We may perhaps form a useful conception of the general nature of certain frequency distributions concerning which the problem of changing the origin is likely to arise by considering a special case for illustration. Assume a frequency curve $y = f(x)$ of considerably less range in the negative direction from some convenient origin than in the positive direction. If d is not less than the range of $y = f(x)$ in the negative direction the probability P_d that a datum taken at random from the distribution will deviate from 0 as much as d in absolute value is the same as the probability P'_{d+h} that the same datum will deviate as much as $d + h$ in absolute value from an origin $0'$ located at a distance h to the left of 0. Thus the problem of changing the origin is likely to arise in a different way from that discussed above where all deviations were taken as positive before the change of origin was made. In the present situation we are likely to have given a distribution

function with the origin at some such point as the arithmetic mean or the mode. Then suppose we seek to move the origin a distance h so as to make the right-hand member of the Tchebycheff inequality

$$P'_{d+h} \leqq M_n/(d+h)^n$$

approach as near as possible to the left-hand member.

This problem differs from the one considered above in that we now seek a value of h which will make

$$V = \Sigma(|x+h|)^n/(d+h)^n, \tag{34}$$

a minimum. If n is an even number the required value of h satisfies the equation

$$dV/dh = [n(d+h)\Sigma(x+h)^{n-1} - n\Sigma(x+h)^n]/(d+h)^{n+1} = 0.$$

When $n=2$ in this equation

$$h = (\Sigma x^2 - d\Sigma x)/(nd - \Sigma x) = (M_2 - dM_1)/(d - M_1). \tag{35}$$

If the first origin is taken at the arithmetic mean, $M_1 = 0$ and (35) becomes

$$h = \mu_2/d. \tag{36}$$

Since h in (36) is positive we know the origin which we seek is to the left of the mean, when the distribution is represented as we have assumed.

To illustrate with a continuous function we take $y = x(10-x)^2$. The centroid of area is at $\bar{x} = 3\frac{1}{3}$. The second moment about the centroid is $\mu_2 = 3.93$. When $d=5$, $h = .786$. The second moment about the new origin is $M_2 = 4.538$, and the deviation is $d+h = 5.786$. We then have $P_d \leqq .16$, and $P'_{d+h} \leqq .13$. We may compare .13 with .16 to note the decrease in the larger member of the inequality due to changing the origin. It is interesting to note that in these illustrations the Tchebycheff inequality gives the best estimate of the value of the probability when the origin is quite near the lower bound of the distribution.

7. *On minimal values of the differences between the members of the inequality.* In the inequality $P \leqq 1/t^n$ it is fairly obvious from very elementary considerations that P and $1/t^n$ each approaches zero as $t \to \infty$. Hence for any distribution of infinite range there exists a deviation from the origin for which the difference between the two members of the inequality is arbitrarily small. But tests with actual distributions indicate that the differences between the two members of the inequality as a rule are not very small except for large values of t. The question may well be raised

as to whether there is a finite value of the variable within the range of its values for which the difference between the two members of the inequality is a minimum. It seems that some distributions are of such a nature that the difference decreases as the deviation d from the origin increases while others have places within the distribution at which the difference is a minimum. In the present section we shall give the results of investigating this question for certain special functions.

To consider this question let us assume a distribution function $f(x)$, continuous and integrable from $-a$ to a, such that $\int_{-a}^{a} f(x)\,dx = 1$. If $x_1 = tM_n^{1/n}$ is a deviation from the mean value of x, we know that

$$P_{x_1} \leqq M_n/x_1^n, \tag{37}$$

where P_{x_1} is the probability that a deviation from the mean is as great as $tM_n^{1/n}$. The inequality (37) may be written

$$\int_{-a}^{x_1} f(x)\,dx + \int_{x_1}^{a} f(x)\,dx \leqq (1/x_1^n) \int_{-a}^{a} x^n f(x)\,dx. \tag{38}$$

We would now ask if there are intervals on x for certain functions $f(x)$ which contain a point at which the difference between the members of (38) is a minimum, and if so what is the nature of the function and where are the intervals.

Let $f(x)$ be positive and symmetrical about the y-axis. Then from (38) we may write the difference which is to be a minimum in the form

$$V = (1/x_1^n) \int_0^a x^n f(x)\,dx - \int_{x_1}^{a} f(x)\,dx. \tag{39}$$

To examine V for a minimum, we have

$$dV/dx_1 = -(n/x_1^{n+1}) \int_0^a x^n f(x)\,dx + f(x_1) = 0,$$

or

$$x_1^{n+1} f(x_1) = n \int_0^a x^n f(x)\,dx. \tag{40}$$

If $f(x)$ is an increasing function, d^2V/dx_1^2 is positive for $n > 0$. Hence a real solution of (40) for x_1 within the interval $M_n^{1/n} \leqq x_1 \leqq a$ corresponds to a minimum for V when $f(x)$ is an increasing function at $x = x_1$. If $f(x)$ is a decreasing function d^2V/dx_1^2 is not necessarily positive and a test for a minimum must be made in each case.

To examine further into the question of the occurrence of interesting minimal values of V, we shall consider certain very special distributions and

seek the value of x_1 that will make V in (39) take a minimal value. In the following cases the ordinary tests for a minimum show that V takes a minimal value at the points indicated.

(a). First assume $f(x) = k$, a constant.

From $2\int_0^a k dx = 1$, we have $k = 1/2a$. We then find $\sigma = a/3^{1/2}$.

From (40), $kx_1^{n+1} = nk\int_0^a x^n dx$ and $x_1^{n+1} = na^{n+1}/(n+1)$.

Let $n = 2$. Then $x_1 = (a/3)(18)^{1/3} > \sigma$.

A minimal value of V exists at this point and we have a very simple function for which the difference between the two members of the inequality attains a minimum at a point fairly near the origin.

(b). Next assume $f(x) = k(1-x^2)$, and $a = 1$.

From $2k\int_0^1 (1-x^2)dx = 1$, $k = 3/4$. We find $\sigma = (.2)^{1/2}$.

There seems to be no point at which the difference between the two members of $P \leqq M_n/x_1^n$ attains a minimum. However, in this case $f(x)$ is a monotonic decreasing functior of x and the inequality of B. H. Camp applies. To test this inequality for a minimum we seek a root of

$$x_1^{n+1}(1+1/n)^n(1-x_1^2) = 2n/(n+1)(n+3),$$

which is between $(.2)^{1/2}$ and 1. Let $n = 2$ and

$$x_1^5 - x_1^3 + 16/135 = 0.$$

This equation has a root between .55 and .56 which corresponds to the minimal value of V.

We have thus found a particular function $f(x)$ for which the closer approximation of Camp would attain a minimum at a value of x which is roughly 11/20 of the value of x at the upper bound of the distribution. It is a fact of some interest that we have found simple functions for which the minimal value of the difference between the members of the inequality exists for values of x_1 in the interval $M_n^{1/n} < x_1 < a$ in case the inequality of Camp is used, but that no minimum exists for the difference if the inequality $P \leqq M_n/x_1^n$ is used.

(c). Take $f(x) = kx^{2b}$ where b is a positive integer $\geqq 1$. From $2\int_0^a kx^{2b}dx = 1$, $k = (2b+1)/2a^{2b+1}$. We find $\sigma = a[(2b+1)/(2b+3)]^{1/2}$.

From (40) when $n = 2b$,

$$x_1^{2b+1}(x_1^{2b}) = 2b\int_0^a x^{4b}dx, \text{ and } x_1 = a[2b/(4b+1)]^{1/(4b+1)}.$$

Now $x > \sigma$ if

$$[2b/(4b+1)]^{1/(4b+1)} > [(2b+1)/(2b+3)]^{\frac{1}{2}}.$$

This inequality holds when b is a positive integer. To prove that the inequality holds for positive integral values of b we write it in the form

$$2b/(4b+1) > [1-2/(2b+3)]^{2b+\frac{1}{2}}.$$

Expanding the right hand member in ascending powers of $2/(2b+3)$ we have an alternating decreasing series and the sum of the first five terms is greater than $[1-2/(2b+3)]^{2b+\frac{1}{2}}$. The sum of these five terms is

$$S = (128b^4 + 64b^3 + 1648b^2 + 1208b + 1137)/24(2b+3)^4.$$

Comparing this expression with $2b/(4b+1)$ we have

$$\frac{2b}{4b+1} - S = \frac{256b^5 + 4224b^4 + 3712b^3 + 3888b^2 - 1868b - 1137}{24(4b+1)(2b+3)^4},$$

which is obviously greater than zero when $b \geqq 1$. The value

$$x_1 = [2b/(4b+1)]^{1/(4b+1)}$$

indicates the point at which V takes a minimal value.

Let us consider next some illustrations with non-symmetric distributions with a range from c to d where c and d are both positive and $c < d$.

(d). Take $f(x) = k/x$. From $k\int_c^d dx/x = 1$, $k\log(d/c) = 1$. We find $M_2 = (d^2 - c^2)/2\log(d/c)$. Instead of equation (39) of the symmetric case, we obtain

$$V = (1/x_1^n)\int_c^d x^n f(x)dx - \int_{x_1}^d f(x)dx,$$

and $dV/dx_1 = 0$ gives $x_1^{n+1}f(x_1) = n\int_c^d x^n f(x)dx$. When $n = 2$, $x_1 = (d^2 - c^2)^{\frac{1}{2}}$. This value is greater than $M_2^{\frac{1}{2}}$ and gives a minimum for V whenever d and c are so selected that

$$d^2 - c^2 > (d^2 - c^2)/2\log(d/c), \text{ or } d/c < e^{\frac{1}{2}}.$$

(e). Take $f(x) = k[x/(1-x)]$ where $c = 0$, and $d = .9$.

From $k\int_c^d [x/(1-x)]dx = 1$, we find $k = 1/1.40259$. We next find $M_2^{\frac{1}{2}} = .73$. From the above value of V when $dV/dx_1 = 0$, we obtain

$$x_1^{n+1}[x_1/(1-x_1)] = n\int_c^d [x^{n+1}/(1-x)]dx,$$

and

$$x_1^{n+2}/(1-x_1) = n\int_0^{.9} [x^{n+1}/(1-x)]dx.$$

When $n = 2$,

$$x_1^4/(1-x_1) = 1.509.$$

This equation has a root between .76 and .77 which corresponds to the minimal value of V.

In conclusion we may say that we have answered the question raised at the beginning of this section for a considerable number of simple functions for which the difference between the two members of the Tchebycheff inequality is a minimum at a point x_1 within the interval from $x = tM_n^{1/n}$ to the upper bound of the distribution. Although certain other functions have been found for which V takes a minimal value, it seems that the cases considered above are sufficient to illustrate the method which may be applied to a function $f(x)$ whenever it is desirable to seek such minimal values.

THE UNIVERSITY OF IOWA.

A Generalization of Eisenstein's Canonical Cubic and Associated Forms.

By Claiborne G. Latimer.

1. Eisenstein treated a canonical cubic, associated forms and reduced associated forms.* The coefficients of the canonical cubic are functions of rational integers β, γ where $(\beta + 3\gamma\rho)(\beta + 3\gamma\rho^2) = q$, ρ being an imaginary cube root of unity and q an arbitrarily chosen prime in the form $3k+1$. He found a necessary and sufficient condition that there exist a reduced form belonging to an integer a and also found the number of such forms.

In this paper we shall consider forms similar to those mentioned above, except that instead of a prime q, we shall use an integer D, which is the product of distinct primes each in the form $3k+1$. For these forms we shall obtain results analogous to those of Eisenstein referred to above, our results being equivalent to his when D is a prime.

The method used here, which is much shorter than Eisenstein's, involves the use of the theory of algebraic numbers. In particular, we shall use the factorable, primitive or *einheit* forms corresponding to the ideals in a cubic field of discriminant D^2 and also certain results obtained by the writer in another paper, hereafter referred to as J.† It will be found that there is a one-to-one correspondence between the ideals of a certain type in the above mentioned field and the reduced associated forms. For the case $q=7$, Nowlan showed that Eisenstein's canonical cubic could be transformed into the norm of the general integral number of a cubic field of discriminant 7^2.‡ All the ideals in this field are principal and hence the corresponding forms are equivalent to the above mentioned norm.

2. Let $D = q_1 \cdot q_2 \cdot \cdots \cdot q_n$ where the q's are distinct primes in the form $3k+1$. Then $q_i = \lambda_i \lambda_i'$ $(i = 1, 2, \cdots, n)$, where λ_i, λ_i' are conjugate imaginary primary primes of $F(\rho)$, and $\prod_{i=1}^{n} \lambda_i = \beta + 3\gamma\rho$, where β, γ are rational integers, $\beta \equiv (-1)^n \pmod 3$. Consider the equation

$$x^3 - 3Dx - (2\beta - 3\gamma)D = 0. \tag{1}$$

* *Journal für Mathematik*, Vol. 28, pp. 289-374.

† "On the Prime Ideals in the General Cubic Galois Field," *American Journal of Mathematics*, Vol. 51 (1929), pp. 295-304.

‡ *Bulletin of the American Mathematical Society*, Vol. 32, p. 380.

$2\beta - 3\gamma$ is prime to D, since $4D = (2\beta - 3\gamma)^2 + 27\gamma^2$, and therefore (1) is irreducible. Let F be the field defined by a root of (1).* By Woronoj's method of determining a basis of the integral numbers of a cubic field,† the following numbers constitute a basis of F:

$$1,\ \theta = (-\xi + \theta_1)/\delta, \qquad \omega_1 = (\xi^2 - 3D + \xi\theta_1 + \theta_1^2)/\delta^2\sigma$$

where δ, σ, ξ are defined as follows. If each of the following conditions is satisfied, $\delta = 3$; otherwise, $\delta = 1$.

$$(2) \qquad 3D \equiv 3 \pmod 9, \qquad \pm(2\beta - 3\gamma)D \equiv -3D + 1 \pmod{27}.$$

σ is the largest integer such that the congruences

$$(3) \quad x^3 - 3Dx - (2\beta - 3\gamma)D \equiv 0 \pmod{\delta^3\sigma^2}, \qquad 3x^2 - 3D \equiv 0 \pmod{\delta^2\sigma},$$

have a common solution, ξ. The discriminant of (1) is $3^6D^2\gamma^2$ and hence the discriminant of F is $3^6D^2\gamma^2/\delta^6\sigma^2$. (2_1) is obviously satisfied and it will be found that (2_2) is also satisfied, the upper or lower sign being read according as n is odd or even. Hence $\delta = 3$. Then σ divides $D\gamma$. D contains no square factor and is prime to $3(2\beta - 3\gamma)$ and therefore by (3), σ is prime to D. If in (3) we set $\delta = 3$, $\sigma = \gamma$ the resulting congruences have the common solution $x = -\beta$. Hence $\delta = 3$, $\sigma = \gamma$ $\xi = -\beta$ and the following numbers constitute a basis of F.

$$1,\ \theta = (\beta + \theta_1)/3, \qquad \omega = (-\beta\theta + \theta^2)/\gamma.$$

Let θ be a root of the cubic equation $g(x) = 0$, the discriminant of which is $D^2\gamma^2$. Then by Theorem 2, *J*, we have the

LEMMA. *A rational prime p, not a divisor of D, is the product of three distinct conjugate prime ideals of F or is a prime of F according as*

$$(4) \qquad g(x) \equiv 0 \pmod p$$

has a solution or has no solution.

3. Let p be a prime, not a divisor of $6\gamma D$, such that $\{p\} = \mathfrak{P}\mathfrak{P}'\mathfrak{P}''$ where the $\mathfrak{P}$'s are distinct prime ideals of F. By the Lemma, (4) has a solution. Since the discriminant of g is a square, by a result due to Woronoj,‡

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* If D is a prime, F is the field defined by one of the three periods of the imaginary Dth roots of unity. See Bachmann, *Die Lehre von der Kreisteilung*, p. 213.

† "Über die ganzen algebraischen Zahlen die von einer Wurzel der Gleichung dritten Grades abhängen." See also J, § 2.

‡ Dickson, *History of the Theory of Numbers*, Vol. 2, pp. 253-4.

it follows that (4) has three solutions, b, b', b'', no two of which are congruent, modulo p. γD is not divisible by p and therefore after adding proper multiples of p to b, b', b'' we may assume

$$g(b)=(b-\theta)(b-\theta')(b-\theta'')=p^m s,\quad g(b')=p^m s',\quad g(b'')=p^m s''$$

where m is an arbitrarily chosen positive integer and s, s', s'' are prime to p. Then $\mathfrak{P}$ divides one and only one of the above factors of $g(b)$, say $b-\theta$. For if it divided two of them, it would divide their difference and then p would divide γD, contrary to hypothesis. $\mathfrak{P}^m$ is therefore the g. c. d. of p^m and $b-\theta$; or $\mathfrak{P}^m=\{p^m, b-\theta\}$. Then

$$\mathfrak{P}^\alpha=\{p^\alpha, b-\theta\},\qquad \mathfrak{P}'^\alpha=\{p^\alpha, b'-\theta\},$$
$$\mathfrak{P}''^\alpha=\{p^\alpha, b''-\theta\}\qquad (\alpha=1,2,\cdots,m).$$

Let $\mathfrak{A}$ be an ideal of F such that it is not divisible by a principal ideal $\{p\}$, where p is a rational prime, and such that $N(\mathfrak{A})=a=p_1^{\lambda_1}\cdot p_2^{\lambda_2}\cdot\cdots\cdot p_k^{\lambda_k}$ where the p's are distinct primes, none of which divide $6\gamma D$. We may then assume

$$\mathfrak{A}=(\mathfrak{P}_1^{\xi_1}\mathfrak{P}_1'^{\eta_1})(\mathfrak{P}_2^{\xi_2}\mathfrak{P}_2'^{\eta_2})\cdots(\mathfrak{P}_k^{\xi_k}\mathfrak{P}_k'^{\eta_k})$$

where $\mathfrak{P}_i\neq\mathfrak{P}_i'$, $N(\mathfrak{P}_i)=N(\mathfrak{P}_i')=p_i$, $\xi_i+\eta_i=\lambda_i$ $(i=1,2,\cdots,k)$. Let $\mathfrak{A}_1$ be an ideal obtained from $\mathfrak{A}$ by replacing every product $\mathfrak{P}_i^{\xi_i}\mathfrak{P}_i'^{\eta_i}$ by one of its conjugates. If an ideal $\mathfrak{G}$ is the product of two ideals $\mathfrak{A}$ and $\mathfrak{A}_1$ as above defined, we shall say that $\mathfrak{G}$ is of type $P(a)$. Let $\mathfrak{P}^\xi\mathfrak{P}'^\eta$ be one of the factors of $\mathfrak{A}$ displayed above and let $\mathfrak{P}'^\xi\mathfrak{P}''^\eta$ be the corresponding factor of $\mathfrak{A}_1$. Assuming that $\xi\geqq\eta$, consider the product,

$$(\mathfrak{P}^\xi\mathfrak{P}'^\eta)(\mathfrak{P}'^\xi\mathfrak{P}''^\eta)=\{p^\eta\}\mathfrak{P}^{\xi-\eta}\mathfrak{P}'^\xi=\{p^\eta,\}\{p^{\xi-\eta}, b-\theta\}\{p^\xi, b'-\theta\},$$

where b, b' are rational integers, incongruent modulo p, determined as above with $m=\xi$. It may be shown that the rational integers in this product are identical with the multiples of $p^{\xi+\eta}=p^\lambda$. If $\eta>\xi$ we obtain the same result in a similar manner. By repetition of this argument we find that the rational integers in $\mathfrak{A}\mathfrak{A}_1$ are identical with the multiples of a. Therefore $\mathfrak{A}\mathfrak{A}_1$ has a canonical basis

$$\mathfrak{A}\mathfrak{A}_1=[a, g+s\theta, h+k\theta+t\omega]$$

where $g,\cdots,t$ are rational integers. We may assume $0\leqq k<s$; $0\leqq g$, $h<a$; and also that $st=a$ since $N(\mathfrak{A}\mathfrak{A}_1)=a^2$.

It is known * that if β_1, β_2, β_3 are integers of F which constitute a basis

* Bachmann, *Allgemeine Arithmetik der Zahlenkörper*, p. 428.

of an ideal $\mathfrak{B}$, then the factorable form

$$U = N(\beta_1 x + \beta_2 y + \beta_3 z)/N(\mathfrak{B})$$

is primitive, i. e. the coefficients are rational integers, the g. c. d. of which is unity. We shall consider later the form corresponding to $\mathfrak{A}\mathfrak{A}_1$; namely,

$$G(x, y, z) = (1/a^2)N[ax + (g + s\theta)y + (p + k\theta + t\omega)z]$$

4. Consider the form

$$\Phi(u, v, w) = u^3 + D(\beta + 3\gamma\rho)(v + w\rho)^3 + D(\beta + 3\gamma\rho^2)(v + w\rho^2) - 3Du(v + w\rho)(v + w\rho^2).$$

A ternary cubic form $f = ax^3 + \cdots$ where $a > 0$, will be said to be *associated with* Φ if its coefficients are rational integers, the g. c. d. of which is unity, and if Φ is transformed into $a^2 f$ by

$$S \qquad \begin{aligned} u &= ax + by + b'z \\ v &= \quad cy + c'z \\ w &= \quad dy + d'z \end{aligned}$$

where $b, \cdots, d'$ are rational integers such that $cd' - c'd = a$. Such a form shall also be said to *belong to* a. If $c' = 0$; $0 \leqq b$, $b' < a$; $0 \leqq d < d'$; f will be said to be a *reduced associated form.** To Φ apply the transformation

$$T_1 \qquad \begin{aligned} u &= \alpha_0 \quad + 2D\alpha_2 \\ v &= \quad\quad 3\gamma\alpha_2 \\ w &= \quad \alpha_1 + \beta\alpha_2. \end{aligned}$$

The resulting form will be found to be $\psi(\alpha_0, \alpha_1, \alpha_2) = N(\alpha_0 + \alpha_1\theta_1 + \alpha_2\theta_1^2)$, where θ_1 is a root of (1).† Setting $\alpha_0 + \alpha_1\theta_1 + \alpha_2\theta_1^2 = x + y\theta + z\omega$, we find that ψ is transformed into $\phi(x, y, z) = N(x + y\theta + z\omega)$ by

$$T_2 \qquad \begin{aligned} \alpha_0 &= x + (1/3)\beta y - (2\beta^2/9\gamma)z \\ \alpha_1 &= \quad (1/3)y - (\beta/9\gamma)z \\ \alpha_2 &= \quad\quad - (1/9\gamma)z. \end{aligned}$$

Consider the form G, defined in § 3. ϕ is transformed into a^2G by the transformation T_3 with the matrix

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* If D is a prime, our definitions of Φ and *reduced associated forms* are identical with Eisenstein's. Our definition of an *associated form* is equivalent to his. See *ibid.*, pp. 297, 333, 334, 318, 296.

† ψ may be obtained from (6), J, by replacing a, b by $-3D$ and $(3\gamma - 2\beta)D$ respectively.

$$\begin{pmatrix} a & g & h \\ 0 & s & k \\ 0 & 0 & t \end{pmatrix}.$$

Apply to G the transformation T_4 with the matrix,

$$\begin{pmatrix} 1 & \xi & \eta \\ 0 & 0 & 3 \\ 0 & 3 & 3\zeta \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & \xi & \eta \\ 0 & 1 & \zeta \\ 0 & 0 & 1 \end{pmatrix}$$

where ξ, η, ζ are arbitrarily chosen rational integers, and let f be the resulting form. G is primitive and T_4 is the product of two transformations as indicated above. The form obtained by the first transformation is primitive since a, the leading coefficient of G, is prime to 3. The second transformation is unitary and a is odd. It may then be shown that f is primitive.*

Φ is transformed into $a^2 f$ by $T_1T_2T_3T_4$. The matrix of this transformation is

$$\begin{pmatrix} a & a\xi + \mu & a\eta + \nu \\ 0 & t & \zeta t \\ 0 & k & s + k\zeta \end{pmatrix}$$

where μ, ν are certain integers independent of ξ, η. By § 3, $st = a$. Hence f is an associated form belonging to a. If we choose ξ, η, ζ so that $0 \leqq a\xi + \mu < a$, $0 \leqq a\eta + \nu < a$, $\zeta = 0$, then f is a reduced associated form.

It follows that a sufficient condition for the existence of a reduced associated form belonging to a, where a is prime to $6\gamma D$, is that there exist an ideal of type $P(a)$; or that every prime ideal factor of a be of the first degree. In the next paragraph we shall show that this condition is also necessary.

5. Let f be a reduced associated form belonging to an integer $a = p_1^{\lambda_1} \cdot p_2^{\lambda_2} \cdot \cdots p_k^{\lambda_k}$ where the p's are distinct primes, none of which divide $6\gamma D$. Then ϕ is transformed into $a^2 f$ by $T_2^{-1}T_1^{-1}S$, where in S, $c' = 0$. The matrix of this transformation is

$$\begin{pmatrix} a & \delta & b' - d'\beta \\ 0 & 3d & 3d' \\ 0 & 3c & 0 \end{pmatrix},$$

where $\delta = b + 2(\beta - 2\gamma)c - d\beta$. Then

$$(3) \qquad f = N[ax + (\delta + 3d\theta + 3c\omega)y + (b' - d'\beta + 3d'\theta)z]/a^2.$$

Let $\mathfrak{B}$ be the g. c. d. of the above coefficients; or

* Eisenstein, *ibid.*, pp. 335-7.

$$\mathfrak{B} = \{a,\ \delta + 3d\theta + 3c\omega,\ b' - d'\beta + 3d'\theta\}.$$

The form $a^2 f/N(\mathfrak{B})$ has rational integral coefficients, the g. c. d. of which is unity.* The same is true of f and hence $N(\mathfrak{B}) = a^2$. $a = cd'$ is prime to 3 and therefore we may determine integers λ, ϵ such that $\lambda ad' + 3\epsilon = 1$. Then

$$a, \qquad \epsilon(\delta + 3d\theta) + c\omega, \qquad \epsilon(b' - d'\beta) + d'\theta$$

belong to $\mathfrak{B}$. These numbers constitute a basis of $\mathfrak{B}$ since $N(\mathfrak{B}) = acd'$; or

$$\mathfrak{B} = [a,\ \ \epsilon(b' - d'\beta) + d'\theta, \qquad \epsilon(\delta + 3d\theta) + c\omega].$$

Consider the factorable, primitive corresponding to $\mathfrak{B}$; namely

$$f'(x', y', z') = (1/a^2)N\{ax' + [\epsilon(b' - d'\beta) + d'\theta]y' + [\epsilon(\delta + 3d\theta) + c\omega]z'\}.$$

Equating the expression in brackets to the corresponding expression in (3) we find that f' is transformed into f by

$$\begin{aligned} x' &= x + \quad \kappa y + \tau z \\ y' &= \quad 3\lambda ad\, y + 3\, z \\ z' &= \quad 3\, y \end{aligned}$$

where κ, τ are rational integers. The determinant of this transformation is -9. We shall now show that $\mathfrak{B}$ is of type $P(a)$.

Suppose $\mathfrak{B}$ were divisible by a prime ideal of the third degree, $\{p\}$. Let $\mathfrak{B} = \{p^n\}\mathfrak{B}_1$, where $\mathfrak{B}_1$ and hence $N(\mathfrak{B}_1)$ is prime to p. By the above basis of $\mathfrak{B}$, c and d' are divisible by p^n. Then a^2 is divisible by p^{4n}, whereas $a^2 = N(\mathfrak{B}) = p^{3n}N(\mathfrak{B}_1)$ where $N(\mathfrak{B}_1)$ is prime to p. We may then assume

$$\mathfrak{B} = (\mathfrak{P}_1^{\xi_1}\mathfrak{P}_1'^{\zeta_1}\mathfrak{P}_1''^{\eta_1})\,(\mathfrak{P}_2^{\xi_2}\mathfrak{P}_2'^{\zeta_2}\mathfrak{P}_2''^{\eta_2}) \cdots (\mathfrak{P}_k^{\xi_k}\mathfrak{P}_k'^{\zeta_k}\mathfrak{P}_k''^{\eta_k})$$

where $\mathfrak{P}_i$, $\mathfrak{P}_i'$, $\mathfrak{P}_i''$ are distinct prime ideals of norm p_i, $\xi_i + \eta_i + \zeta_i = 2\lambda_i$, $\zeta_i \geqq \xi_i \geqq \eta_i \geqq 0$ $(i = 1, 2, \cdots, k)$. Consider the product $\mathfrak{P}_1^{\xi_1}\mathfrak{P}_1'^{\zeta_1}\mathfrak{P}_1''^{\eta_1}$. Dropping subscripts, we have

$$\mathfrak{P}^{\xi}\mathfrak{P}'^{\zeta}\mathfrak{P}''^{\eta} = \{p^{\eta}\}\mathfrak{M},\ \mathfrak{M} = \mathfrak{P}^{\xi-\eta}\mathfrak{P}'^{\zeta-\eta} = \{p^{\xi-\eta}, b - \theta\}\{p^{\zeta-\eta}, b' - \theta\}$$

where b, b' are rational integers, determined as in § 2 with $m = 2\lambda$. The rational integers in $\mathfrak{B}$ are identical with the multiples of a and hence those in $\mathfrak{M}$ are the multiples of $p^{\lambda-\eta}$. Since $N(\mathfrak{M}) = p^{2\lambda-3\eta}$, it follows that $\mathfrak{M}$ has the canonical basis

$$\mathfrak{M} = [p^{\lambda-\eta},\ R + S\theta,\ T + U\theta + V\omega]$$

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* Bachmann, *Allgemeine Arithmetik der Zahlenkörper*, pp. 425-6.

where $R, \cdots, V$ are positive integers and $SV = p^{\lambda-2\eta}$. $\mathfrak{M}$ contains $(b-\theta)(b-\theta') = bb' + (\beta - b - b')\theta + \gamma\omega$ and since a is prime to γ it follows that $V = 1$. $\mathfrak{M}$ contains $p^{\xi-\eta}(b-b')$ and hence it contains $p^{\xi-\eta}$. Therefore $\lambda \leqq \zeta$, or $\lambda \geqq \xi + \eta$. $\mathfrak{M}$ also contains $p^{\xi-\eta}(b'-\theta)$ and therefore $S = p^{\lambda-2\eta}$ divides $p^{\xi-\eta}$; or $\lambda \leqq \xi + \eta$. Hence $\lambda = \xi + \eta = \zeta$ and

$$\mathfrak{P}^{\xi}\mathfrak{P}'^{\xi}\mathfrak{P}''^{\eta} = (\mathfrak{P}^{\xi}\mathfrak{P}'^{\eta})\,(\mathfrak{P}'^{\xi}\mathfrak{P}''^{\eta})$$

Repeating this argument, we find that $\mathfrak{B}$ is of type $P(a)$.

If $\alpha_1, \alpha_2, \alpha_3$ and $\beta_1, \beta_2, \beta_3$ are two bases of an ideal $\mathfrak{B}$, then $\alpha_i = \sum_j c_{ij}\beta_j$ $(i = 1, 2, 3)$ where c_{ij} are rational integers and the determinant $|\, c_{ij} \,| = \pm 1$. It follows that the primitive factorable forms corresponding to these bases are equivalent. We have therefore

THEOREM 1. *Let F be the cubic field defined by a root of* (1) *and let $\mathfrak{G}$ be an ideal of F of type $P(a)$, where a is prime to $6\gamma D$. If $\alpha_1, \alpha_2, \alpha_3$ are integers of F which constitute a basis of $\mathfrak{G}$, then the primitive, factorable form*

$$(1/a^2)N(\alpha_1 x + \alpha_2 y + \alpha_3 z)$$

may be transformed into a reduced associated form belonging to a by a transformation, T, with rational integral coefficients, of determinant ± 9.

Conversely, if f is such a reduced associated form, where a is prime to $6\gamma D$, there exists an ideal of type $P(a)$ such that every primitive factorable form corresponding to this ideal may be transformed into f by such a transformation T.

COROLLARY. *If a is a positive integer, prime to $6\gamma D$, there exists a reduced associated form belonging to a if and only if every rational prime factor of a is the norm of an ideal of F.**

By the corollary and Theorem 4, *J*, we have

THEOREM 2. *If a is a positive integer, prime to $6\gamma D$, a sufficient condition that there exist a reduced associated form belonging to a is that every prime factor of a be a cubic residue of D. If D is a prime, this condition is also necessary.*†

6. In this paragraph we shall show that there is a one-to-one correspondence between the ideals of type $P(a)$ and the reduced associated forms

* Theorem 3, J, gives a necessary and sufficient condition that a rational prime be such a norm.

† Eisenstein obtained this result for D a prime; *ibid.*, p. 352.

belonging to a. We shall then find the number of such ideals and hence the number of such forms.

In the preceding paragraph we saw that to every reduced form there corresponded an ideal $\mathfrak{B}$ of type $P(a)$. Let f_1, f_2 be two reduced forms belonging to a and suppose they correspond to the same ideal $\mathfrak{B}$. Then Φ is transformed into $a^2 f_1$, $a^2 f_2$ by transformations with the following matrices:

$$\begin{pmatrix} a & b_1 & b_1' \\ 0 & c_1 & 0 \\ 0 & d_1 & d_1' \end{pmatrix}, \quad \begin{pmatrix} a & b_2 & b_2' \\ 0 & c_2 & 0 \\ 0 & d_2 & d_2' \end{pmatrix},$$

where $0 \leqq b_i, b_i' < a$; $0 \leqq d_i < d_i'$; $c_i d_i' = a$ $(i = 1, 2)$. We have

$$\begin{aligned} \mathfrak{B} &= [a, \epsilon_1(b_1' - \beta d_1') + d_1'\theta, \epsilon_1(\delta_1 + 3d_1\theta) + c_1\omega] \\ &= [a, \epsilon_2(b_2' - \beta d_2') + d_2'\theta, \epsilon_2(\delta_2 + 3d_2\theta) + c_2\omega] \end{aligned}$$

where the ϵ_i are determined by $\lambda_i a d_i' + 3\epsilon_i = 1$ and $\delta_i = b_i + 2(\beta - 3\gamma)c_i - d_i\beta$. From the above sets of basal numbers it follows that $c_1 = c_2$, $d_1' = d_2'$ and hence we may assume $\epsilon_1 = \epsilon_2$. Since the numbers of each set may expressed in terms of the other, it is readily shown that the forms f_1 and f_2 are identical. Hence the number of reduced forms belonging to a is at most equal to the number of ideals of type $P(a)$.

From every ideal of type $P(a)$ we obtain a reduced form belonging to a. Suppose we obtain the same form f from two ideals. Let G and G' be the corresponding primitive, factorable forms, as defined at the end of § 3. Then G and G' are transformed into f by transformations T_4 and T_4' respectively, with matrices

$$\begin{pmatrix} 1 & \xi & \eta \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & \xi' & \eta' \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{pmatrix}.$$

Then G' is transformed into G by a transformation with the matrix

$$\begin{pmatrix} 1 & (\eta' - \eta)/3 & (\xi' - \xi)/3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The coefficients of G and G' are rational integers and hence the same is true of the coefficients of this transformation. It follows that the ideals corresponding to G and G' are equal. Hence the number of reduced forms belonging to a is at least equal to the number of ideals of type $P(a)$. Therefore the number of such ideals is exactly equal to the number of reduced forms.

Consider the number of ideals of type $P(a)$, assuming that there is one. If $a = p^\lambda$, p a prime, the distinct ideals of type $P(a)$ are

$$(\mathfrak{P}^{\lambda-i}\mathfrak{P}'^{i})(\mathfrak{P}'^{\lambda-i}\mathfrak{P}''^{i}), \qquad (i = 1, 2, \cdots, \lambda).$$

where the $\mathfrak{P}$'s are distinct primes of norm p, and the 2λ ideals obtained from these by replacing $\mathfrak{P}, \mathfrak{P}', \mathfrak{P}''$ by $\mathfrak{P}', \mathfrak{P}'', \mathfrak{P}$ and by $\mathfrak{P}'', \mathfrak{P}, \mathfrak{P}'$ respectively. Hence there are exactly 3λ such ideals. If $a = p_1^{\lambda_1} \cdot p_2^{\lambda_2} \cdot \cdots p_k^{\lambda_k}$, where the p's are distinct primes, we obtain $3^k\lambda_1 \cdot \lambda_2 \cdot \cdots \lambda_k$ ideals of the required type.

THEOREM 3. *If $a = p_1^{\lambda_1} \cdot p_2^{\lambda_2} \cdot \cdots p_k^{\lambda_k}$, where the p's are distinct primes, none of which divide $6\gamma D$, and if there is one reduced form belonging to a, there are exactly*

$$3^k\lambda_1 \cdot \lambda_2 \cdot \cdots \lambda_k$$

*such reduced forms.**

7. In § 2 we defined $\beta + 3\gamma\rho$ as a product of primary primes. We may obtain $2^{n-1} = t$ such products, $\beta_i + 3\gamma_i\rho$ $(i = 1, 2, \cdots, t)$ such that the norm of each is D and no two of them are conjugates. For each of these products we obtain as in § 2 a field of discriminant D^2 and also a corresponding Φ_i as defined in § 4. The Lemma and Theorems 1, 2, 3 may be applied to each of these Φ_i and the corresponding field F_i $(i = 1, 2, \cdots, t)$. We shall now prove.

THEOREM 4. *If a is a positive integer prime to $6\gamma_1 \cdot \gamma_2 \cdot \cdots \gamma_t \cdot D$, then for every form Φ_i $(i = 1, 2, \cdots, t)$ there exists a reduced associated form belonging to a if and only if a is a cubic residue of D.*

In view of Theorem 2, it suffices to prove the necessary condition when $n > 1$. Suppose a is an integer, prime to $6\gamma_1 \cdot \gamma_2 \cdot \cdots \gamma_t \cdot D$, such that for every Φ_i there is a reduced associated form belonging to a, and suppose p is a prime factor of a. By the corollary of Theorem 1, p is the product of three conjugate ideals in each of the fields F_i $(i = 1, 2, \cdots, t)$. Let $\beta_1 + 3\gamma_1\rho = \prod_{i=1}^{n} \lambda_i$, $\beta_2 + 3\gamma_2\rho = \lambda_1' \prod_{i=2}^{n} \lambda_i$ where the λ_i are primary primes of $F(\rho)$ and λ_1' is the conjugate of λ_1. Let F_1 and F_2 be the corresponding fields, and let $p = \pi \cdot \pi'$ or $p = \pi$ where π is a primary prime of $F(\rho)$. Since p is a composite in F_1 and in F_2 by Theorem 3, J, and (c) of Lemma 2, J, we have

$$\prod_{i=1}^{n} \left[\frac{\pi}{\lambda_i}\right] = \prod_{i=1}^{n} \left[\frac{\pi}{\lambda_i'}\right], \quad \left[\frac{\pi}{\lambda_1'}\right] \prod_{i=2}^{n} \left[\frac{\pi}{\lambda_i}\right] = \left[\frac{\pi}{\lambda_1}\right] \prod_{i=2}^{n} \left[\frac{\pi}{\lambda_i'}\right],$$

* Eisenstein obtained this result for D a prime. See *ibid.*, p. 352.

where the brackets are Eisenstein's symbols for the cubic character of π with respect to the corresponding primes. Therefore

$$\left[\frac{\pi}{\lambda_1}\right] = \left[\frac{\pi}{\lambda_1'}\right].$$

Squaring both sides, we have

$$\left[\frac{\pi'}{\lambda_1'}\right] = \left[\frac{\pi}{\lambda_1}\right].$$

If $p = \pi \cdot \pi'$ by multiplying these equations we obtain

$$\left[\frac{p}{\lambda_1}\right] = \left[\frac{p}{\lambda_1'}\right],$$

which is also valid if $p = \pi$. But

$$\left[\frac{p}{\lambda_1}\right]^2 = \left[\frac{p}{\lambda_1'}\right].$$

Hence each of these symbols is unity; or p is a cubic residue of each of the non-associated prime factors of $\lambda_1\lambda_1' = q_1$. Repetition of this argument shows that p is a cubic residue of $q_1 q_2 \cdots q_n = D$. This proves the theorem.

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Quadruples of Space Curves.*

By A. R. Jerbert.

1. *Introduction.* Several papers have appeared recently which employ the notion of configurations whose elements are in one-to-one-correspondence. These papers are: "Ruled Surfaces with their Generators in one-to-one Correspondence," by E. P. Lane; † "Triads of Ruled Surfaces," by A. F. Carpenter,‡ and "Triads of Plane Curves," by the writer.§

The present paper is a logical continuation of the last one cited. It bears, however, a certain relation to Professor Lane's paper. This consists in the fact that in both papers the analytic basis is a system of four, linear, homogeneous, differential equations of the first order. In each case the given system of equations defines, to within a projectivity, four space curves.

From this point on the viewpoints are divergent. In Professor Lane's paper the four points are paired. One pair defines the ruling of one surface, while the other pair defines the corresponding ruling for a second surface. The curves serve merely as directrix curves for the surfaces and as such they are replaced, at an early stage, by more suitable curves, namely, the "intersector curves." ¶

In the present paper the curves given by the defining system are regarded as the entity under consideration. To this end, only transformations which preserve the identity of the curves are permitted. These transformations are, the change of parameter and the transformation which multiplies the homogeneous coördinates by an arbitrary factor.

2. *Defining System.* Let us suppose that the four homogeneous coördinates of a point x_1 are given as functions of a parameter t; that is, $x_1^{(i)} = f_i(t)$, $(i = 1, 2, 3, 4)$. As the parameter t varies, the point x_1 describes a curve C_{x_1}. In the same fashion we shall assume that three other points, x_2, x_3 and x_4, have their coördinates given as functions of the same parameter t.

As t varies, the four points describe the space curves C_{x_1}, C_{x_2}, C_{x_3}, and C_{x_4} which are in point correspondence by virtue of their expression in terms of

* Read before the American Mathematical Society, June 2, 1928.

† *Transactions of the American Mathematical Society*, Vol. 25 (1923), p. 281.

‡ *Transactions of the American Mathematical Society*, Vol. 29 (1927), p. 254.

§ *American Journal of Mathematics*, Vol. 51 (1929), p. 99.

¶ *Loc. cit.*, p. 287.

the common parameter t. All the functions involved will be assumed to be continuous and differentiable except at most for isolated values of the parameter.

We proceed to find for the four curves a defining system of the form,

$$x_i' = \sum_{j=1}^{4} a_{ij} x_j \quad (i = 1, 2, 3, 4), \tag{1}$$

where differentiation is with respect to t. The problem reduces to computing the coefficients a_{ij}. To solve for the a_{1i} $(i = 1, 2, 3, 4)$ substitute in the first of equations (1), in turn, the first, second, third, and fourth coördinates of the points x_1, x_2, x_3, x_4. The resulting system of four equations can be solved for the a_{1i} providing that the determinant

$$\Delta \equiv | x_1^{(1)}, x_2^{(2)}, x_3^{(3)}, x_4^{(4)} | \tag{2}$$

is not identically zero.

In the same manner the remaining coefficients can be found provided always that the determinant (2) does not vanish identically. This assumption we shall make. Geometrically it amounts to asserting that the correspondence between the curves is such that the corresponding points are not coplanar, except at most, for isolated values of t.

If in place of the four points $x_1, \cdots x_4$ we employ four new points obtained from the latter by the most general projectivity we find that they give rise to the same defining system (1). System (1), therefore, defines the configuration of four associated space curves only to within a projectivity and is accordingly suitable for the investigation of the projective differential properties of such a configuration.

3. *Canonical System.* As stated in the introduction, the object of this paper is to study the mutual relation of the curves defined by (1). With this in view the only permissible transformations are the change of parameter and the multiplicative transformation on the homogeneous coördinates. We shall employ the latter to obtain a canonical form for system (1). To this end let us put

$$x_i = \lambda_i \bar{x}_i \quad (i = 1, 2, 3, 4), \tag{3}$$

where the λ_i $(i = 1, 2, 3, 4)$, are arbitrary functions of the parameter t. If we substitute these expressions in (1) we obtain after transposition and division,

$$\bar{x}_i' = \sum_{j=1}^{4} (a_{ij}\lambda_j - \delta_{ij}\lambda_j')\bar{x}_j/\lambda_i \quad (i = 1, 2, 3, 4). \tag{4}$$

From (4) it is apparent that we can choose the arbitrary proportionality

factors in such a way as to render the coefficients in the principal diagonal equal to zero. For this purpose we shall take

$$\lambda_i = k_i e \exp \int a_{ii}\delta t, \qquad (i = 1, 2, 3, 4).$$

We obtain, accordingly, the defining system

$$\begin{aligned}
\bar{x}_1' &= \bar{a}_{12}\bar{x}_2 + \bar{a}_{13}\bar{x}_3 + \bar{a}_{14}\bar{x}_4, \\
\bar{x}_2' &= \bar{a}_{21}\bar{x}_1 + \bar{a}_{23}\bar{x}_3 + \bar{a}_{34}\bar{x}_4, \\
\bar{x}_3' &= \bar{a}_{31}\bar{x}_1 + \bar{a}_{32}\bar{x}_2 + \bar{a}_{34}\bar{x}_4, \\
\bar{x}_4' &= \bar{a}_{41}\bar{x}_1 + \bar{a}_{42}\bar{x}_2 + \bar{a}_{43}\bar{x}_3 ,
\end{aligned} \tag{5}$$

where

$$\bar{a}_{ij} = a_{ij}\lambda_j/\lambda_i. \tag{6}$$

One other "permissible" transformation remains and that is the change of parameter. It is readily verified that the effect of such a transformation is to multiply the coefficients in the defining system by a factor. Since we can not utilize the parameter transformation to any advantage, we shall pay no further heed to it. For the purposes of this paper, therefore, defining system (5) will serve as a canonical system.

4. *Invariants and Covariants.* From (6) we see that such products of the coefficients as $a_{ij}\,a_{ji}$ are invariants under the λ-transformation. The coefficient a_{ij} carries with it a factor λ_j/λ_i whereas a_{ji} involves the factor λ_i/λ_j. The invariancy of $a_{ij}\,a_{ji}$ is rendered apparent if we think of an inner subscript as cancelling a like outer subscript. With this in mind, such a product as $a_{23}\,a_{34}\,a_{42}$ is obviously invariant.

As a corollary to the last remark we note that if in the determinant of the coefficients a_{ij} (equation 5) we replace the diagonal terms by 1's and expand by the ordinary rule all the terms in the expansion are invariants.

With respect to covariants we note similarly that such an expression as $a_{41}\,x_3 - a_{31}\,x_4$ carries with it, on being transformed, a factor $\lambda_1/\lambda_3\,\lambda_4$ and is therefore a relative covariant. The following invariants are listed and named for convenience of reference.

$$\begin{aligned}
&A_{12} = a_{12}\,a_{21}, \quad A_{13} = a_{13}\,a_{31}, \quad A_{14} = a_{14}\,a_{41}, \\
&A_{23} = a_{23}\,a_{32}, \quad A_{24} = a_{24}\,a_{42}, \quad A_{34} = a_{34}\,a_{43}, \\
&M_1 = a_{23}\,a_{34}\,a_{42}, \qquad N_1 = a_{32}\,a_{43}\,a_{24}, \\
&M_2 = a_{13}\,a_{34}\,a_{41}, \qquad N_2 = a_{31}\,a_{14}\,a_{43}, \\
&M_3 = a_{21}\,a_{14}\,a_{42}, \qquad N_3 = a_{12}\,a_{24}\,a_{41}, \\
&M_4 = a_{12}\,a_{23}\,a_{31}, \qquad N_4 = a_{21}\,a_{13}\,a_{32},
\end{aligned} \tag{7}$$

We can characterize the canonical system (5) geometrically if we recall that the point x_1' is a point on the tangent to the curve C_{x_1} at the point x_1.

The canonical system implies that we have chosen the proportionality factors (the λ_i) in such a fashion that x_1' is the point where the tangent to C_{x_1} meets the opposite face of the tetrahedron. Similar statements hold for each of the vertices of the tetrahedron.

Since the work that follows will be based entirely upon the canonical system (5), no ambiguity will occur if we write the coefficients which appear in (5) without the bars.

5. *Dual Considerations.* It will be recalled that the assumption was made at the outset, that the correspondence between the four curves was to be such that corresponding points should not, in general, be coplanar. It follows that if we join corresponding points by straight lines a non-degenerate tetrahedron is formed. Since the parameter t can take a singly infinite number of values we see that the study of the four associated curves is equivalent to the study of a one parameter family of tetrahedrons.

Since a tetrahedron is determined by its face planes as well as by its vertices it will prove useful to find a defining system for the coördinates of these planes. If we denote by u_i the plane opposite the vertex x_i it follows that the coördinates of u_i, i. e. $u_i^{(k)}\,(k = 1, 2, 3, 4)$, are proportional to the cofactors of $x_i^{(k)}\,(k = 1, 2, 3, 4)$, in the determinant Δ (equation 2). We shall assume for the coördinates of u_i the cofactors themselves. By direct computation we find by the aid of equations (5) the defining system

$$
(8)\qquad \begin{aligned}
u_1' &= \quad\;\; - a_{21}u_2 - a_{31}u_3 - a_{41}u_4,\\
u_2' &= -a_{12}u_1 \quad\;\; - a_{32}u_2 - a_{42}u_4,\\
u_3' &= -a_{13}u_1 - a_{23}u_2 \quad\;\; - a_{43}u_4,\\
u_4' &= -a_{14}u_1 - a_{24}u_2 - a_{34}u_3 \quad\;\; .
\end{aligned}
$$

It will be observed that the defining system (8) like (5) is in canonical form. If the computation for (8) had been based upon the uncanonical system (1) we would have obtained in place of (8) a system like it but with the diagonal terms restored.

The geometrical interpretation of equations (8) is dual to that of equations (5). Equation $(8)_1$ states that the plane u_1' passes through the intersection of the planes u_2, u_3, and u_4. Similar statements hold for equations $(8)_2$, $(8)_3$, and $(8)_4$.

6. *Characteristic Lines.* As the parameter t varies each face of the tetrahedron gives rise to a one parameter family of planes which envelop an edge of regression.

If, in any one of the face planes, we wish to determine the line which is tangent to the edge of regression, in other words the characteristic line, we find that this is most readily accomplished by the aid of equations (8).

For this purpose we have only to recall that the characteristic line represents the intersection of the plane with its immediate successor. In the dual case we determine the join of the point x_1 and its immediate successor as the line joining x_1 and x_1'. We conclude, therefore, that the characteristic line in the plane u_i may be found as the intersection of the planes u_i and u_i'.

It proves convenient at this point to introduce a local coördinate system. We are, so to speak, already provided with a suitable reference tetrahedron. There remains merely the choice of unit point. We shall choose the latter in such a manner that an expression of the form $y_i\, x_i$ $(i = 1, 2, 3, 4)$ shall represent the point whose local coördinates are y_i $(i = 1, 2, 3, 4)$.

With this choice of coördinate system the planes u_1 and u_1' become in point coördinates, the planes

$$y_1 = 0,\ a_{21}\, y_2 + a_{31}\, y_3 + a_{41}\, y_4 = 0. \tag{9}$$

Equations (9) taken simultaneously, define the characteristic line of the plane u_1 $(y_1 = 0)$. The characteristic line so defined, intersects the edges of the tetrahedron lying in its plane in three points given by the expressions

$$a_{31}\, x_2 - a_{21}\, x_3,\ a_{41}\, x_3 - a_{31}\, x_4,\ a_{41}\, x_2 - a_{21}\, x_4. \tag{10}$$

In a similar way each characteristic line determines a set of three points, twelve in all, two for each edge of the tetrahedron. The points (10) are manifestly covariant as we should be led to expect from their geometrical character. Before proceeding with the properties of the characteristic points we shall find it advisable to introduce other considerations.

7. *Tangent Planes.* Each edge $x_i\, x_j$ of the tetrahedron generates a ruled surface as the parameter t varies and the points x_i, x_j describe their respective curves. The plane determined by the points x_i, x_j, and $(x_i + \lambda x_j)'$, is tangent to the ruled surface $(x_i\ x_j)$ at the point $x_i + \lambda x_j$. For the six ruled surfaces we obtain the following tangent planes:

$$\begin{aligned}
&(x_1\, x_2),\ (a_{13} + \lambda a_{23})y_4 - (a_{14} + \lambda a_{24})y_3 = 0,\\
&(x_1\, x_3),\ (a_{12} + \lambda a_{32})y_4 - (a_{14} + \lambda a_{34})y_2 = 0,\\
&(x_1\, x_4),\ (a_{13} + \lambda a_{43})y_2 - (a_{12} + \lambda a_{42})y_3 = 0,\\
&(x_2\, x_3),\ (a_{21} + \lambda a_{31})y_4 - (a_{24} + \lambda a_{34})y_1 = 0,\\
&(x_2\, x_4),\ (a_{21} + \lambda a_{41})y_3 - (a_{23} + \lambda a_{43})y_1 = 0,\\
&(x_3\, x_4),\ (a_{31} + \lambda a_{41})y_2 - (a_{32} + \lambda a_{42})y_1 = 0.
\end{aligned} \tag{11}$$

$(11)_4$ reduces to $y_1 = 0$, if we take for λ the value $-a_{21}/a_{31}$. In other words, the point $(10)_1$ is the point on the ruled surface $(x_2\, x_3)$ at which the tangent plane is identical with the plane of the points x_2, x_3, and x_4. In simi-

lar fashion we find the points on $(x_3\,x_4)$ and $(x_4\,x_2)$ at which the tangent planes reduce to $x_1 = 0$. They prove to be the other characteristic points given by (10).

The fact thus disclosed can be stated for a single ruled surface.

THEOREM 1. *If a plane be passed through each ruling of a ruled surface there results a one parameter family of planes which possesses the following property. Each plane is tangent to the ruled surface at that point, of the ruling through which it is passed, in which the latter is met by the characteristic line of the plane in question.*

On the other hand, if we wish to find the equation of the tangent plane to $x_1\,x_2$ at x_1 we have merely to set λ in (11) equal to zero, thus obtaining the plane

$$a_{13}\,y_4 - a_{14}\,y_3 = 0. \tag{12}$$

The plane (12 meets the opposite edge x_3x_4 in a point whose expression is

$$a_{13}x_3 + a_{14}x_4. \tag{13}$$

In the same manner we determine the points in which the tangent planes to $x_1\,x_4$ and $x_1\,x_2$ at x_1 meet the opposite edges. We obtain in this way three points in all. They are

$$a_{13}\,x_3 + a_{14}\,x_4,\ a_{12}\,x_2 + a_{13}\,x_3,\ a_{12}\,x_2 + a_{14}\,x_4. \tag{14}$$

From the manner in which the expressions (14) are formed it is clear that each of the corresponding points is collinear with the point x_1' (equation 5), and the opposite vertex. We state this result as

THEOREM 2. *In a one parameter family of tetrahedrons the planes constructed at any vertex, tangent to the ruled surfaces intersecting in the curve generated by this vertex, intersect the edges of the opposite face in a set of three points. The lines joining these points to the opposite vertices in this face are concurrent in the point at which the tangent to the curve described by the vertex in question meets the opposite face.*

8. *Space Quadrilaterals.* Under (14) we note that there are 12 such points—two on each edge of the tetrahedron. A number of theorems can be stated which are concerned with the relations that may exist between the latter set of points and those we have denoted as characteristic points. The following relations may be readily verified by reference to equations (5), (9), and two others similar to (9) for the x_4 characteristic.

$$
\begin{array}{lll}
 & A_{12} + A_{13} = 0, & (1^3\,4\,),\ x_1\,x_4\,x_1' \text{ meets } x_2\,x_3 \text{ in } x_1 \text{ char,} \\
(15) & a_{12}\,a_{24} + a_{13}\,a_{34} = 0, & (1^2\,4^2),\ x_1\,x_4\,x_1' \quad " \quad x_2\,x_3 \quad " \quad x_4 \quad " \ , \\
 & A_{24} + A_{34} = 0, & (1\ 4^3),\ x_1\,x_4\,x_4' \quad " \quad x_2\,x_3 \quad " \quad x_4 \quad " \ , \\
 & a_{42}\,a_{21} + a_{31}\,a_{43} = 0, & (4^2\,1^2),\ x_1\,x_4\,x_4' \quad " \quad x_2\,x_3 \quad " \quad x_1 \quad " \ ,
\end{array}
$$

Equation $(15)_1$ states that the vanishing of the invariant $A_{12} + A_{13}$ (equations 7) is the necessary and sufficient condition that the tangent plane, at x_1, to the ruled surface generated by the edge $x_1\,x_4$ shall meet the corresponding ruling of the opposite ruled surface in the point in which that ruling is met by the characteristic line of the plane u_1 $(y_1 = 0)$. We have described this line as the x_1 characteristic because it lies opposite the vertex x_1. The parenthesis indicates which subscripts appear in the geometrical statement and how often each one occurs.

The point in which $x_1\,x_4\,x_2'$ meets $x_2\,x_3$ is actually the point $(14)_2$ so that $(15)_1$ simply states under what condition the last named point coincides with the characteristic point $(10)_1$. We have preferred to state the relation as in $(15)_1$. By varying the subscripts similar relations can be written for the remaining edges of the tetrahedron.

If we remove from a tetrahedron a pair of opposite sides, the remaining edges constitute a unicursal four-line or space quadrilateral. Since we may remove any one of three different pairs of sides it follows that the tetrahedron may be thought of as giving rise to three distinct space quadrilaterals.

If we consider now the parentheses $(1^2\,2^2)$, $(2^2\,3^2)$, $(3^2\,4^2)$, and $(4^2\,1^2)$ we see that the edges they refer to form a space quadrilateral. It is to be noted also that the parentheses indicate a direction along this quadrilateral, namely, the direction pursued in going successively to the vertices x_1, x_2, x_3, and x_4.

The geometrical situations associated with each of the parentheses are given in statement $(15)_4$ and similar statements obtainable from the latter by rotating subscripts. It is readily verified that the accompanying invariant equalities are such that any three imply a fourth. We have accordingly,

THEOREM 3. *If any one of a one parameter family of space quadrilaterals be traced unicursally and if at three successive vertices the plane tangent at the vertex to the ruled surface generated by the ensuing edge meets the opposite edge in the point where the latter is met by the characteristic line corresponding to the successor vertex, then for the fourth vertex the property still persists.*

There is an important distinction between the theorem just stated and theorems 1 and 2. The last mentioned theorems were of general validity;

i. e., they required no hypothesis except that implied in the configuration. Theorem 3, on the other hand, stated that if a certain property recurred three times it would necessarily occur a fourth time. We can indicate this feature of the theorem by the fraction (3/4).

Most of the theorems of this paper will be like theorem 3. As to type they will be either (1/2), (2/3), (3/4), or (5/6). The possibility of each type existing is clear if we keep in mind the fact that there are four vertices, six edges, three edges in a plane, three edges on a vertex, while opposite edges occur in pairs.

The following considerations lead to a theorem of the (3/4) type. From any vertex an infinite number of tangent lines can be drawn to a ruled surface. We shall put our attention upon just one of these, namely, that one which meets the ruled surface in the ruling which corresponds to the vertex in question; *i. e.*, the ruling shall be given by that value of the parameter t which determines the vertex. From a given vertex three such lines can be drawn, one to each of the three ruled surfaces generated by the remaining three vertices taken in pairs. We shall speak of these as the "instantaneous" tangents from the vertex in question.

If we inquire under what conditions the "instantaneous" tangents from any vertex x_i are coplaner we find as necessary and sufficient condition

$$M_i = N_i, \text{ (equations 7).} \tag{16}$$

Of the four equalities (16) only three are independent. We have accordingly,

THEOREM 4. *If from three of the vertices of a variable tetrahedron the "instantaneous" tangents drawn to the ruled surfaces generated by the remaining edges are coplanar then for the fourth vertex the same property holds.*

9. *Developables.* The condition that the edge $x_i x_j$ shall generate a developable is simply the condition that the points x_i, x_j, x_i', and x_j' shall be coplaner. Applying this test to each edge in turn we find the following conditions for developables,

$$\begin{aligned} &(x_1 x_4)\, a_{12} a_{43} = a_{13} a_{42}, \quad (x_1 x_3)\, a_{32} a_{14} = a_{34} a_{12}, \\ &(x_2 x_3)\, a_{21} a_{34} = a_{31} a_{24}, \quad (x_2 x_4)\, a_{41} a_{23} = a_{21} a_{43}, \\ &(x_1 x_2)\, a_{13} a_{24} = a_{14} a_{23}, \quad (x_3 x_4)\, a_{42} a_{31} = a_{32} a_{41}. \end{aligned} \tag{17}$$

It is readily verified that only five of the equalities (17) are independent. We have, accordingly,

THEOREM 5. *If five of the edges of a variable tetrahedron generate developables the sixth edge also generates a developable.*

It is also readily verified that if the edges meeting in a vertex generate developables equation (16) holds. We have therefore

THEOREM 6. *If the three edges meeting in a vertex generate developables the instantaneous tangents from this vertex to the ruled surfaces generated by the opposite edges are coplanar.*

10. *Fixed Point and Plane.* If the coördinates of a point P referred to a fixed tetrahedron are P_i $(i = 1, 2, 3, 4)$, the necessary and sufficient conditions that the point P shall be fixed are,

$$P_i' = \phi P_i, \quad (i = 1, 2, 3, 4). \tag{18}$$

Let us suppose that the coördinates of a point P referred to our variable tetrahedron as reference system, are a, b, c, and d. To find the conditions which make P a fixed point we must, in order to avail ourselves of (18), express the coördinates of P in terms of a fixed tetrahedron. For this purpose the obvious choice is to take as fixed tetrahedron the one in which the coördinates of the vertices x_1, x_2, x_3, x_4, of the variable tetrahedron are expressed. With this understanding

$$P \equiv a\,x_1 + b\,x_2 + c\,x_3 + d\,x_4. \tag{19}$$

If we substitute the expression (19) in (18) and treat the result as an identity in x_1, x_2, x_3, and x_4 we obtain the following conditions upon a, b, c, and d:

$$\begin{aligned} a' &= \quad \phi a - a_{21}b - a_{31}c - a_{41}d, \\ b' &= -a_{12}a + \phi b - a_{32}c - a_{42}d, \\ c' &= -a_{13}a - a_{23}b + \phi c - a_{43}d, \\ d' &= -a_{14}a - a_{24}b - a_{34}c + \phi d. \end{aligned} \tag{20}$$

If we multiply the coördinates a, b, c, and d by the factor $\exp \int \phi dt$, we obtain equations (20) with the diagonal terms missing; that is, we obtain, except for a change in notation, the adjoint system (8). We can speak of either (20) or the equivalent canonical system as the "geometric adjoint" * of system (1) or the equivalent canonical system (5). In similar fashion we find that the coördinates of a fixed plane, after multiplication by a properly chosen factor, satisfy equations (5). We can summarize these results in

THEOREM 7. *The coördinates of a point which is fixed in space but referred to a variable tetrahedon, satisfy the system of equations which define*

* Cf. A. L. Nelson, The Conditions for a Fixed Point in Projective Differential Geometry, *Bulletin of the American Mathematical Society*, Vol. 32 (1926), p. 166.

the face planes of the tetrahedron. The coördinates of a fixed plane, on the other hand, satisfy the defining system for the vertices of the tetrahedron.

The following considerations lead to an equivalent statement of Theorem 6. To this end it is necessary to recall the peculiar manner in which the defining system (1) was set up. The determinant Δ (equation 2) involves 16 functions arranged in a four-by-four array. In computing the coefficients in (1) we interpreted the elements in the i'th column of Δ as the coördinates of a point x_i. To state the same fact in a slightly dfferent form, suppose we are given the system of equations (1). A set of simultaneous solutions x_1, x_2, x_3 and x_4 is not interpreted as a point. On the contrary, if we have four such sets of solutions, the four values of x_i define the point x_i.

Looking back at equations (20) we see that in this instance it is precisely the simultaneous solutions which we interpret as the coördinates of a fixed plane referred to the variable tetrahedron. We have accordingly,

THEOREM 7_1. *If in a square array of 16 functions, with determinant different from zero, we interpret the elements in each column as the coördinates of a point, we obtain four points. As the parameter in which the functions are expressed, varies, the four points together with their joins generate a variable tetrahedron. If the elements in a given row are interpreted as the coördinates of a plane referred to the variable tetrahedron, then this plane is fixed in space. The four rows, accordingly, define four fixed planes.*

11. *Duo-flecnode Points.* With the variation of the parameter t, a pair of opposite edges of the tetrahedron will generate two ruled surfaces which are in one-to-one line correspondence. In connection with such a configuration, Professor Lane * has introduced the notion of straight line intersectors of the four skew lines made up of the corresponding rulings of the two surfaces together with their successors.

The two points in which a ruling of either surface is met by the intersectors in limiting position we shall denote as the "duo-flecnode points." The term flecnode is suggested by the similar situation in ordinary ruled surface theory. The prefix was added to distinguish from another case to be mentioned shortly.

Since any point on the edge $x_i x_j$ is given by the expression $x_i + \lambda x_j$, the determination of the flecnode points consists in determining appropriate values for λ. The following quadratics determine the values of λ for the edges indicated

* *Loc. cit.*, p. 291.

$$
\begin{aligned}
&(x_3\,x_4)-(a_{13}\,a_{41}+a_{23}\,a_{42})\lambda^2+(P+Q)\lambda+(a_{14}\,a_{31}+a_{32}\,a_{24})=0,\\
&(x_1\,x_2)-(a_{24}\,a_{41}+a_{31}\,a_{23})\lambda^2+(P-Q)\lambda+(a_{13}\,a_{32}+a_{14}\,a_{42})=0,\\
(21)\quad &(x_1\,x_4)-(a_{21}\,a_{42}+a_{21}\,a_{43})\lambda^2+(R+P)\lambda+(a_{24}\,a_{12}+a_{13}\,a_{34})=0,\\
&(x_2\,x_3)-(a_{12}\,a_{31}+a_{34}\,a_{42})\lambda^2+(R-P)\lambda+(a_{13}\,a_{21}+a_{24}\,a_{43})=0,\\
&(x_1\,x_3)-(a_{21}\,a_{32}+a_{34}\,a_{41})\lambda^2+(R-Q)\lambda+(a_{23}\,a_{12}+a_{14}\,a_{43})=0,\\
&(x_2\,x_4)-(a_{12}\,a_{41}+a_{32}\,a_{43})\lambda^2+(R+Q)\lambda+(a_{14}\,a_{21}+a_{23}\,a_{34})=0,
\end{aligned}
$$

where

$$P=A_{24}-A_{13},\ Q=A_{14}-A_{23},\ R=A_{34}-A_{12}.$$

By observing the form of the coefficient of λ in equations (21) we see that the following theorems are true:

THEOREM 8. *If on two of the edges of a variable tetrahedron, the duo-flecnode points are harmonic with respect to adjacent vertices it follows that the property holds for the third edge coplanar with the first two.*

THEOREM 9. *If on one, and only one, line of each of two pairs of opposite edges, the flecnode points are harmonic with respect to adjacent vertices it follows that on one, and only one, line of the remaining pair of edges are the flecnode points harmonic with respect to adjacent vertices.*

12. *Tri-flecnode Points.* If we have two ruled surfaces with generators in one-to-one correspondence we can obtain another set of points similar to the ones we have designated as duo-flecnode points.* This time we make a selection of four skew lines by choosing on the first surface a ruling together with two successor rulings. The corresponding ruling on the second surface constitutes a fourth line. The two straight line intersectors of the four lines take a limiting position as the successor rulings on the first surface approach coincidence with the ruling itself.

The points where the ruling of the first surface is met by the intersectors in limiting position we shall designate as the tri-flecnode points on that ruling. On the other hand, the points where the ruling of the second surface is met by these same intersectors will be designated as the uno-flecnode points on that ruling. In the equations below, the quadratic in λ determines the tri-flecnode points on $(x_1\,x_4)$, while the similar quadratic in μ determines the corresponding uno-flecnode points on $(x_2\,x_3)$.

$$
\begin{aligned}
(22)\quad &\lambda^2(a_{42}C_4-a_{43}B_4)+\lambda(a_{42}C_1-a_{13}B_4+a_{12}C_4-a_{43}B_1)+(a_{12}C_1-a_{13}B_1)=0,\\
&\mu^2(a_{12}B_4-a_{42}B_1)+\mu(\ {}_{42}C_1-a_{13}B_4-a_{12}C_4+a_{43}B_1)+(a_{13}C_4-a_{43}C_1)=0.
\end{aligned}
$$

* E. P. Lane, *loc. cit.*, p. 291.

The expressions A_i, B_i, C_i, D_i are defined by the equation,

$$x_i'' = A_i x_1 + B_i x_2 + C_i x_3 + D_i x_4.$$

By permuting subscripts similar quadratics can be written for the remaining edges.

13. *Tri-flecnode Points Indeterminate.* For the tri-flecnode points on $(x_1 x_4)$ to be indeterminate it is necessary and sufficient that the coefficients of λ^2, λ^1, and λ^0 in $(22)_1$ shall be zero. We must have,

$$(23) \qquad \begin{aligned} &a_{42} C_4 = a_{43} B_4, \quad a_{12} C_1 = a_{13} B_1, \\ &a_{42} C_1 + a_{12} C_4 - a_{43} B_1 + a_{13} B_4. \end{aligned}$$

The third of equations (23) when taken in conjunction with the other two leads to two distinct possibilities. The first term in the left member may be equated to either the first or second term in the right member. We have, therefore, as a consequence of (23), either the following set of conditions

$$(24) \quad a_{42} C_4 = a_{43} B_4, \; a_{12} C_1 = a_{13} B_1, \; a_{42} C_1 = a_{13} B_4, \; a_{12} C_4 = a_{43} B_1,$$

or

$$(25) \quad a_{42} C_4 = a_{43} B_4, \; a_{12} C_1 = a_{13} B_1, \; a_{12} C_4 = a_{13} B_4, \; a_{42} C_1 = a_{43} B_1.$$

From equations (24) we readily find that the coefficients of μ^2, μ^1, μ^0 in $(22)_2$ are zero. In other words, if equations (24) hold, the edge $x_2 x_3$ lies on the osculating quadric to the ruled surface generated by the opposite edge $x_1 x_4$.

On the other hand, equations (25) are the necessary and sufficient conditions that the edge $x_1 x_4$ should generate a plane. In fact, if we insist that the osculating plane to the curve traced by x_1 shall be identical with the corresponding one for the curve traced by x_4, we obtain the last two of equations (25) together with a third equation which can be obtained from the first two of equations (25).

If, then, equations (25) hold the edge $x_1 x_4$ generates a plane. The plane thus determined will intersect the opposite edge in a point. All the lines lying in the plane will meet the edge $x_1 x_4$ and its two successor rulings as well as the edge $x_2 x_3$. The indeterminateness of the tri-flecnode points is obvious in this case.

If equations (24) hold, the points determined by $(22)_2$ are also indeterminate so that the ruling $x_1 x_4$, together with its two successor rulings and the opposite ruling $x_2 x_3$ are all in quadric position. We can summarize this section as,

THEOREM 9. *If two ruled surfaces are in one-to-one line correspondence, indeterminateness of the tri-flecnode curves on one of the two surfaces*

*implies one of two things, the surface on which the curves are indeterminate is a plane, or else the second surface belongs to the flecnode congruence of the first.**

From equations (25) we can readily derive equation $(17)_1$, which is the condition that $(x_1\, x_4)$ should generate a developable. By rotating letters and subscripts we obtain from (25) the conditions that each edge in turn shall generate a plane. In each case only three of the four equations are independent. We shall choose as independent conditions the first two in each set together with the condition for developability. We already know that the latter conditions are so related that any five imply the sixth. It also turns out that the first two in each set are so related that if they hold for five edges they necessarily hold for the sixth edge. We have then,

THEOREM 10. *If five of the edges of a variable tetrahedron generate planes, the sixth edge also generates a plane.*

CONCLUSION.

We have considered in this paper a one parameter family of tetrahedrons. The various parts of the configuration have been seen to be interdependent in such a fashion that properties of a certain portion of the configuration implied like properties for the rest of the configuration.

Only a few of the simplest properties have been considered.

If we wish to apply the results of this paper to the variable reference tetrahedron as it occurs in various branches of Projective Differential Geometry we need merely to have for the vertices of the tetrahedron such a defining system as (1) or (5).

If we turn to the theory of space curves, we find that equations (5), p. 239 of Wilczynski's *Proj. Diff. Geom.*, are just such a system as we require. In fact, if, as is usually the case, p_1 is set equal to zero, equations (5) are in what we have designated as the canonical form.

In general surface theory the points y, y_u, y_v, and y_{uv} frequently constitute the reference tetrahedron. If the parameter u alone varies a one parameter family of tetrahedrons is obtained. There is no difficulty in writing down a canonical system for these points.

A succeeding paper will extend these results to four-space.

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* Cf. Wilczynski, *Projective Differential Geometry*, p. 175.

On the Moduli of Algebraic Functions Possessing a Given Monodromie Group.

By Oscar Zariski.

1. It is known that, given a transitive group G of degree n generated by a set of substitutions $S_1, S_2, \cdots, S_\omega$, such that $S_1 S_2 \cdots S_\omega = 1$, there exist infinite algebraic functions y of x of order n possessing G as monodromie group and having ω branch points $\alpha_1, \alpha_2, \cdots, \alpha_\omega$, which can be assigned arbitrarily, and such that for a given system of loops in the x plane the substitution on the branches of y at α_i is exactly S_i. By letting the position of the branch points α_i vary arbitrarily, we obtain a *complete continuous system,* which we will call Σ, of algebraic functions, which depend on $\omega - 3$ essential parameters—the cross-ratios of the set (α). All functions of Σ are of the same order n and have the same monodromie group G and the same genus p. The value of p depends only on the assigned substitutions S_i, viz., if S_i is composed of μ_i cycles of orders $\nu_{1i}, \nu_{2i}, \cdots, \nu_{\mu_i i}$ respectively, then

$$2n + 2p - 2 = \sum_{i=1}^{\omega} \left(\sum_{j=1}^{\mu_i} \nu_{ji} - \mu_i\right). \tag{1}$$

We now ask: *Under what conditions will the functions of Σ be of general moduli?* By saying that an n-valued algebraic function y is of general moduli, we mean that the algebraic curve ϕ, defined in the invariantive sense of Algebraic Geometry, and on which y corresponds to a definite linear involution g_n^1, is of general moduli. Evidently a *necessary* condition, that the functions y of Σ be of general moduli, is that the above number $\omega - 3$ of essential parameters should not be less than the number $3p - 3$ of the birational moduli of the general curve of genus p. Hence, necessarily

$$\omega \geqq 3p. \tag{2}$$

If (2) holds, will it be possible to affirm that the generic function y of Σ is of general moduli? There is at least one case in which this is not necessarily true, namely when G is an imprimitive group. In fact, in this case, any function y of Σ defines on the corresponding curve ϕ a g_n^1 *composed of an involution.* If this involution is irrational and $p > 1$, then it is well known that ϕ, or y, is of special moduli. The following example may serve as an illustration: Let $\omega = 6$, and let

$$S_1 = S_2 = S_3 = S_4 = (13)\ (24)\,; \quad S_5 = S_6 = (12). \tag{3}$$

The functions y of Σ are of genus 2 and the number of essential parameters on which they depend is 3, the same as that of the birational moduli in the case $p = 2$. However, we prove, that the functions y are of special moduli. In fact, the group G of degree 4 and of order 8 generated by the substitutions (3), is imprimitive, the sets of imprimitivity being [1, 2]; [3, 4]. Hence the g_4^1 is composed of an involution γ_2^1. If ψ is an algebraic curve in one-to-one correspondence with the sets of the γ_2^1 on ϕ, then to the g_4^1 on ϕ there corresponds on ψ a g_2^1, and hence a two-valued algebraic function z, whose branch points are evidently among the branch points of y. Now, the substitutions S_1, S_2, S_3, S_4 permute the sets of imprimitivity, while S_5 and S_6 leave invariant each set. Hence z, a 2-valued algebraic function with 4 branch points α_1, α_2, α_3, α_4, is of genus 1. It follows, that the γ_2^1 on ϕ is an elliptic involution, and that therefore the functions y of Σ are of special moduli.

Excluding the obvious case of imprimitive groups, the above formulated question constitutes one of the most general problems, which arise in connection with the theory of Galois as applied to Algebraic Geometry. The affirmative answer, i. e. the theorem that if G is a primitive group and if (2) holds, then the functions of Σ are of general moduli, is probably correct, although at present we are not able to prove it rigorously. The theorem which we establish in this paper (section 3), and which generalizes a theorem stated by Severi for the case in which the assigned substitutions S_i are transpositions,* may be considered as the first step toward the solution of the problem. It gives an important geometric criterion, which reduces this group problem to a problem of the theory of virtually complete series on an algebraic curve.

2. Our problem may be considered from a slightly different point of view, which we are going to explain. On a given curve ϕ of genus p and of general moduli there exist, for a sufficiently large given value of n, an infinite class of linear involutions g_n^1. The monodromie group of *a generic* g_n^1 of the class, i. e. the monodromie group of the corresponding n-valued algebraic function, is the total group. But for special g_n^1's of the class the group may well reduce to a subgroup of the total group. This will generally happen when and only when each multiple point of the g_n^1 is of order > 2, so that the Jacobian set of the g_n^1 does not contain simple points. We call a transitive group G of degree n *non-special of genus* p, if G is the monodromie group of *some* g_n^1 on a curve ϕ of genus p *and of general moduli.* Otherwise we say that G *is special of genus* p. All transitive groups of a given degree are

* F. Severi, "Sul Teorema Di Esistenza Di Riemann," *Rendiconti del Circolo Matematico di Palermo*, Vol. 46 (1922).

thus divided, *with respect to a given value p*, into two classes, special and non-special groups of genus p. It follows from the definition that, if a group G of degree n is special of genus p, then no matter how we specialize the g_n^1 on a curve of genus p of general moduli, by letting its Jacobian set acquire multiple points, we never obtain a g_n^1 possessing G as monodromie group. Our problem amounts to finding a criterion which should enable us to decide whether a given transitive primitive group G is special or non-special of genus p.

Let, as in section 1, a set of substitutions $S_1, S_2, \cdots, S_\omega$ be introduced, which generate G and such that $S_1 S_2 \cdots S_\omega = 1$. We say then, that we have *a representation of genus p and of rank ω of the group G*, where p is defined by (1). A group G is certainly special of genus p, if in *any representation* of genus p of G, the rank ω is less than $3p$. The most important example of such groups is given by the primitive solvable groups. The author showed elsewhere * that if $p > 6$, then in any representation of genus p of a primitive solvable group the rank ω is less than $3p$, and thus deduced the conclusion that an algebraic equation of genus $p > 6$ and of general moduli cannot be solved parametrically in radicals.

On the other hand, if a transitive primitive group G does admit a representation of genus p and of rank $\omega \geqq 3p$, it does not yet follow, although it seems probable (see section 1), that G is non-special of genus p. If this were true, we would have the following remarkable fact: that the special groups of a given genus can be completely characterized by an arithmetical condition.

3. If the assigned substitutions S_i are transpositions, so that G is the total group of degree n, it is well known that G is special of genus p, if and only if $n < p/2 + 1$. A theorem due to Severi (see quoted paper) says that the most general curve of genus p containing a g_n^1 depends on $3p - 3 - i$ moduli, where i is the *index of speciality* of the complete two-fold series $| 2g_n |$. The purpose of this paper is to show that the method employed by Severi can still be successfully employed in the case in which the assigned substitutions are (not transpositions, but) arbitrary. We thus obtain a theorem similar to that of Severi, in which, however, the ordinary complete series $| 2g_n |$ is replaced by a conveniently defined *virtually complete series* $|| 2g_n ||$. In order to be able to state our result, we anticipate some of the definitions and statements of the later sections.

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* Sull'impossibilità di risolvere parametricamente per radicali un'equazione algebrica $f(x, y) = 0$ di genere $p > 6$ a moduli generali, *Rendiconti della R. Accademia dei Lincei* (6), Vol. 3 (1926).

Let y be a generic function of Σ, and let g_n^1 be the linear involution on the corresponding algebraic curve ϕ. Let $\Gamma_1, \Gamma_2, \cdots, \Gamma_\omega$ be the sets of the g_n^1 which contain multiple points of the involution. Then (see section 1) Γ_i contains one multiple point P_{1i} of order ν_{1i}, one multiple point P_{2i} of order $\nu_{2i}, \cdots$, one multiple point $P_{\mu_i i}$ of order $\nu_{\mu_i i}$ and $n - \nu_{1i} - \nu_{2i} - \cdots \nu_{\mu_i i}$ simple points. Consider the ω sets

$$(4) \qquad (P_{1i}^{\nu_{1i}-1}, \quad P_{2i}^{\nu_{2i}-1}, \cdots, \quad P_{\mu_i i}^{\nu_{\mu_i i}-1}).$$

Given a linear series g_m^r on ϕ, the above sets are said to be *neutral sets* of the series, if each set imposes one condition only on the sets of the g_m^r constrained to contain it, i. e. if the sets of the g_m^r which contain one point of one of the sets (4), say P_{1i}, consequently contain each point P_{ji} of this set to the multiplicity $\nu_{ji} - 1$.

If the g_m^r is not contained in a larger series of the same order, possessing the same ω neutral sets, then the series is said to be virtually complete.

A set A of m points of which none coincides with any of the points P_{ji}, is contained in a unique virtually complete series $\| A \| = \| g_m \|$.

For reasons which will be explained later, the *index of speciality* i of a virtually complete series g_m^r is defined as follows:

$$i = r - [m - p - \sum_{i=1}^{\omega} (\sum_{j=1}^{\mu_i} \nu_{ji} - \mu_i)] = r - [\omega - 3p + 2 + m - 2n].$$

A virtually complete series is *special* or *non-special* according as i is > 0 or $= 0$.

In particular, a virtually complete series of order $2n$ is a series $g_{2n}^{\omega-3p+2+i}$, where i is the index of speciality of the series.

We can now announce our result as follows:

Let Γ and Γ' be two arbitrary sets of the g_n^1, which corresponds to a generic function y of Σ, and let $\| g_{2n} \| = \| \Gamma + \Gamma' \|$ be the virtually complete series defined by the set $\Gamma + \Gamma'$. The functions of Σ depend on $3p - 3 - i$ moduli, where i is the index of speciality of the series $\| g_{2n} \|$.

In particular, we have:

The necessary and sufficient condition that the functions of Σ be of general moduli (and hence, that the group G be non-special of genus p), is that the above virtually complete series $\| g_{2n} \|$ should be non-special.

4. Let y be a generic function of the system Σ, and let ϕ be the algebraic curve on which y corresponds to a g_n^1. By means of this g_n^1 and of a generic

g_m^1 on ϕ, we transform ϕ into a plane algebraic curve C of order $m+n$ with two fixed multiple points A and B of orders m and n respectively. The curve C possesses other variable multiple points equivalent to $(m-1)(n-1)-p$ ordinary double points. The linear series g_n^1 and g_m^1 are cut out on C by the lines of the pencils A and B respectively. By letting the ω branch points α of the function y and the g_m^1 on ϕ vary, we obtain a continuous system of curves C, which we will denote by K. *We say that if m is taken sufficiently large, the system K is irreducible, and that the generic curve C of K possesses ordinary double points only, and that moreover none of the ω tangents to C through A passes through a double point of C.*

To see this, let us take $m \geqq 2p+2$, and let us consider a generic complete series g_m^{m-p} on ϕ. We imagine the sets of the series represented by the points of a projective space S_{m-p}, in which the lines are the images of g_m^1's contained in the g_m^{m-p}. As an immediate consequence of the theorem of Riemann-Roch it follows, that the g_m^{m-p} has no fixed points, and that its residual series with respect to any 3 (distinct or coincident) points, is a g_{m-3}^{m-p-3}.

We consider in S_{m-p} the following loci:

1. The locus V_1 of points, images of the sets of the series g_m^{m-p} which contain a variable triple of points of ϕ, belonging to a set of the g_n^1.

2. The locus V_2 of points, images of the sets of the g_m^{m-p}, which contain a pair of (distinct or coincident) points belonging to one of the ω sets of the g_n^1, which contain multiple points of this last series.

The loci V_1 and V_2 consist of a finite number of varieties, and by the foregoing remark each of these varieties is of dimensions $\leq m-p-2$. Hence, it is sufficient to consider a generic line in S_{m-p} which does not meet the loci V_1 and V_2, in order to have a g_m^1 on ϕ which together with the g_n^1 gives rise to a curve C of K, satisfying the required conditions.

That the system K is irreducible, follows immediately from the fact that both the system Σ and the variety of all g_m^1's on a curve of genus p are irreducible.

To evaluate the *dimension* of K, we observe that, when the ω tangents to C through A are assigned, the projective correspondence between the sets of the g_n^1 and the lines of the pencil A is uniquely determined,* while there are still ∞^3 ways of setting up a projective correspondence between the sets of the g_m^1 and the lines of the pencil B. Since the dimension of the variety of the linear involutions g_m^1 on a curve of genus p is $2m-p-2$, it follows that K is of dimension

* We suppose that $\omega \geqq 3$, since, if $\omega = 2$, the curves C are rational.

$$r = 2m - p - 2 + 3 + \omega = 2m - p + \omega + 1. \tag{5}$$

The peculiarity of the system K consists in the fact that each of the ω tangents through A to a generic curve C of K is, in general, a multiple tangent, having with C one or more contacts of order $\geqq 1$. From the foregoing considerations it follows that the points of contact of any of the ω tangents are *distinct simple points* of C. Thus, if we denote the tangent corresponding to the branch point α_i by the same letter α_i, we have that α_i has with C one contact of order ν_{1i} at a simple point P_{1i}, one contact of order ν_{2i} at a simple point $P_{2i}, \cdots$, one contact of order $\nu_{\mu_i i}$ at a simple point $P_{\mu_i i}$.

5. Following the method of Severi, we consider the $\sigma = 2m + 2p - 2$ tangents through B to a generic curve C. Let us denote them by λ. If of the σ tangents λ, $\sigma - i$ $(i \geqq 0)$ only can be assigned arbitrarily, then the curves of K depend on $3p - 3 - i$ moduli. Let C_0 be a generic but fixed curve of K and let us consider the variety of curves of K which touch the tangents λ. Let H be that *irreducible* part of this variety which contains C_0. Since, by hypothesis, the sets λ depend on $\sigma - i$ arbitrary parameters, and since the dimension of K is $2m - p + \omega + 1$, it follows that the *dimension* of H is

$$r' = (2m - p + \omega + 1) - (2m + 2p - 2 - i) = \omega - 3p + 3 + i, \tag{6}$$

and that the characteristic series of H on C_0 is of dimension

$$r' - 1 = \omega - 3p + 2 + i. \tag{7}$$

In order to characterize the above series, we prove the following:

LEMMA. *If in an irreducible continuous system $\{D\}$ of plane algebraic curves, the generic curve D has μ contacts of orders $\nu_1, \nu_2, \cdots, \nu_\mu$ at simple points $P_1, P_2, \cdots, P_\mu$ with a variable line α passing through a fixed point A, then the characteristic series of the system on D possesses $[P_1^{\nu_1 - 1}, P_2^{\nu_2 - 1}, \cdots, P_\mu^{\nu_\mu - 1}]$ as a neutral set of points.*

Let D_0 be a generic fixed curve of $\{D\}$. To simplify the proof, we may suppose that the point A is at infinity on the y-axis, and that the multiple tangent a to D_0 coincides with the y-axis. We consider a generic algebraic system ∞^1 of curves D, containing D_0. The equation of a generic curve D of the system is of the form

$$f(x, y, t) = (x + \xi)\phi(x + \xi, y, t) + (y + b_1)^{\nu_1}(y + b_2)^{\nu_2} \cdots (y + b_\mu)^{\nu_\mu}\psi(y, t) = 0,$$

where ξ and b_i are functions of t. We suppose that D_0 corresponds to $t=0$, so that the equation of D_0 reduces to

$$f(x, y, 0) = x\phi_0(x, y) + (y + b_1^0)^{\nu_1}(y + b_2^0)^{\nu_2} \cdots (y + b_\mu^0)^{\nu_\mu}\psi_0(y),$$

where

$$\phi_0(x, y) = \phi(x, y, 0), \quad b_i^0 = \lim_{t=0} \cdot b_i, \quad \psi_0(y) = \psi(y, 0),$$

and

$$\phi_0(0, -b_i^0) \neq 0, \qquad \psi_0(-b_i^0) \neq 0 \qquad (i = 1, 2, \cdots, \omega).$$

The characteristic group on D_0 is cut out by the curve

$$(8) \quad [\partial f(x, y, t)/\partial t]_{t=0} = [\phi_0(x, y) + x(\partial\phi_0/\partial x)]\,(d\xi/dt)_0 + x(\partial\phi/\partial t)_0 + (y + b_1^0)^{\nu_1 - 1}(y + b_2^0)^{\nu_2 - 1} \cdots (y + b_\mu^0)^{\nu_\mu - 1}\bar{\psi}(y) = 0,$$

where

$$(9) \quad \bar{\psi}(y) = (y + b_1^0)(y + b_2^0) \cdots (y + b_\mu^0) \times \{\psi(y, 0)[\nu_1/(y + b_1^0) + \nu_2/(y + b_2^0) + \cdots + \nu_\mu/(y + b_\mu^0)] + (\partial\psi/\partial t)_0\},$$

and where $(\partial\phi/\partial t) = [\partial\phi(x + \xi, y, t)/\partial t]$ is evaluated by keeping ξ constant. Suppose now that the characteristic group contains the point $x = 0$, $y = -b_1^0$. Then, since $\phi_0(0, -b_1^0) \neq 0$, it follows from (8) and (9) that $(d\xi/dt)_0 = 0$, and hence that the curve (8) passes through each point $x = 0$, $y = -b_i^0$ and has at that point a $(\nu_i - 1)$-fold contact with the y-axis, q. e. d.

By this lemma it follows that the characteristic series of H on C_0 possesses ω neutral sets:

$$(4') \qquad [P_{1i}^{\nu_{1i}-1}, \quad P_{2i}^{\nu_{2i}-1}, \cdots, \quad P_{\mu_i i}^{\nu_{\mu_i i}-1}], \qquad (i = 1, 2, \cdots, \omega).$$

Moreover, the curves of H, infinitely near to C_0, pass through the double points of C_0 and through the σ points of contact of the tangents λ with C_0. Hence, as in the case considered by Severi, the above characteristic series is of order $2n$ and is totally contained in the two-fold of the series g_n^1, cut out by the lines of the pencil A.

The characteristic series of H on C_0 can be thought of as *a virtually complete series of index of speciality i.* To show this, we must give a general definition of a virtually complete series, and we must also explain what is meant by a special virtually complete series and by the index of speciality of such a series.* This is done in the next section.

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* The general theory of virtual series was first developed by Nöther in "Über die Schnittpunktsysteme einer algebraischen Curve mit nicht-adjungirten Curven," *Mathematische Annalen*, Vol. 15 (1879), p. 507-528. In this paper Nöther uses as

6. A linear series $g_m{}^r$ on a curve ϕ, with ω neutral sets

$$(10) \qquad [P_{1i}{}^{k_{1i}},\ P_{2i}{}^{k_{2i}},\cdots,\ P_{\mu_i i}{}^{k_{\mu_i i}}], \qquad (i=1,2,\cdots,\omega)$$

is *virtually complete,* if it is not contained in a series of the same order and greater dimension, possessing the same ω sets (10) as neutral sets. A virtually complete series will be denoted by $\| g_m \|$.

A set G_m of m points of ϕ, none of which coincides with a point P_{ji} of the above sets, belongs to a unique virtually complete series. Due to this theorem, this series can be denoted by $\| G_m \|$.

To prove this, let $g_m{}^r$ $(r \geqq m-p)$ be the complete series to which the set G_m belongs. In the representative space S_r, the points of which correspond to the ∞^r sets of the $g_m{}^r$, let the point O correspond to the set G_m, and let S_{r_i} be the linear space in S_r the points of which correspond to the sets of the $g_m{}^r$, which contain the points $P_{1i}, P_{2i},\cdots,P_{\mu_i i}$ to the multiplicities $k_{1i}, k_{2i},\cdots,k_{\mu_i i}$ respectively. By hypothesis, the spaces S_{r_i} $(i=1, 2,\cdots,\omega)$ do not contain the point O. Any linear series $g_m{}^\sigma$, containing G_m, and having the above ω neutral sets of points, is represented in S_r by a linear space, which evidently belongs to each of the spaces S_{r_i+1} determined by O and S_{r_i}. Hence any such series is contained in a unique series of the greatest dimension, which is represented in S_r by the space $S\rho$ — intersection of the ω spaces S_{r_i+1}. *This series $g_m{}^\rho$, for which the above ω sets* (10) *are neutral sets of points, is the virtually complete series, uniquely determined by the set G_m.*

The virtually complete series $g_m{}^\rho$, or $\| G_m \|$, is said to be non-special, if the following conditions are satisfied:

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a starting point for his theory the definition of non-adjoint, or virtually adjoint curves, and derives two distinct types of virtually complete series on a given curve f. Series of the first type are cut out on f by complete linear systems of curves, subject only to the condition of being virtually adjoint curves of f. Series of the second type are cut out on f by complete linear systems of virtually adjoint curves, which are "*gleichsingulär*," this being the term which Nöther uses in order to express the fact that the curves of the system in addition satisfy certain contact conditions at each multiple point of f. It clearly appears, although it is not pointed out explicitly by Nöther, that these virtual series of the second type possess neutral sets of points. In our exposition we base the definition of virtual series on this property. This procedure seems to us to be the more simple and to lead rapidly to the fundamental theorems of the theory, avoiding the complicated analysis relative to the behaviour of "*gleichsingulär*" virtually-adjoint curves at the singular points of the curve f.

For a treatment of virtual series with neutral *pairs*, see F. Severi, *Vorlesungen über algebraische Geometrie*, Leipzig, Teubner (1921), Anhang F., p. 349-353.

1.° *The ordinary complete series* g_m^r, *or* $|\, G_m \,|$, *is non-special, and hence* $r = m - p$.

2.° *The points* P_{ji} *of the* i*-th neutral set, taken to the corresponding multiplicities* k_{ji}, *impose* $k_{1i} + k_{2i} + \cdots + k_{\mu_i i}$ *independent conditions on the sets of the series* g_m^r, *constrained to contain them;* hence,

$$r_i = m - p - k_{1i} - k_{2i} - \cdots - k_{\mu_i i} \geqq 0. \tag{11}$$

3.° *The* ω *spaces* S_{r_i+1} *are linearly independent and so intersect in a space of dimension*

$$\rho = (r_1 + 1) + (r_2 + 1) + \cdots + (r_\omega + 1) - (\omega - 1)r. \tag{12}$$

Putting

$$\sum_{i=1}^{\omega} \sum_{j=1}^{\mu_i} k_{ji} = k,$$

and taking into account (11), (12) becomes

$$\rho = m - p - k + \omega. \tag{13}$$

Conversely, if the dimension ρ of a virtually complete series $\| G_m \|$ is given by (13), the conditions 1°, 2° and 3° all hold, and hence the series is non-special.

If one, at least, of the above conditions is not satisfied, the virtually complete series $\| G_m \|$ is *special*. The dimension of a special virtually complete series g_m^ρ is given by the formula

$$\rho = m - p - k + \omega + i, \tag{14}$$

where i *is* > 0. *The number* i *is called the index of speciality of the series.**

* The above definitions suggest that it is possible, as in the case of neutral pairs, to consider a virtually complete series as the limit of an ordinary complete series on a variable curve of genus $P = p + k - \omega$. It will be noticed however, that if $k > 2\omega$, i. e. if not all the neutral sets are neutral pairs, the upper limit $2p - 2 + k$ [See (15)] of the order of a special series $|| g_m ||$ is less than $2P - 2$, and that hence a special virtually complete series of the highest order is *not* the limit of the cannonical series g_{2P-2} on a variable curve of genus P. For instance, if we consider series having one neutral triple ($k = 3$), the special series of the highest order are series g_{2p+1}, and these are limits of series contained in the canonical series g_{2p+2}^{p+1} on the variable curve of genus $p + 2$. If a curve of genus p is represented by a plane curve f_0 of order m with d ordinary double points and one ordinary triple point O, any of the above series g_{2p+1} is cut out by a system of (*virtually adjoint*) curves of order $m - 3$, defined by the condition of passing *simply* through the double points of f and through the point O and of having at O a fixed tangent. If f is a variable curve of genus $p + 2$ with $d + 1$ double points, which tends to f_0, the curves of the above system are limits of the adjoint curves of f, passing through a fixed point O', which, as f tends to f_0, approaches O on a definite direction.

It is easily shown that the order m of a special virtually complete series g_m^ρ cannot surpass $2p-2+k$. In fact, the residual series of the g_m^ρ with respect to the neutral sets is of order $m-k$ and of dimension $\rho-\omega = m-k-p+i$. Hence the residual series is special in the ordinary sense, and therefore

$$m \leqq 2p-2+k. \tag{15}$$

Without going into the details of the proof, we observe that *there exist special virtually complete series of the highest order* $2p-2+k$. But, while in the case of neutral pairs of (distinct or coincident) points, there exists only one such series, viz., *the virtual canonical series,* in any other case this is not true any more. If $k > 2\omega$, there exist infinite special virtual series of the highest order, each of which may be considered as *a* virtual canonical series. However, the following property still holds: *a special virtual series is always contained in one of the virtual canonical series.*

7. The characteristic series of the system K on C_0 is a $g^{2m-p+\omega}_{2n+2m+2p-2}$ (see sec. 4), possessing the ω neutral sets (4) of sec. 3. We have in this case

$$k = \sum_{i=1}^{\omega} \sum_{j=1}^{\mu_i} (\nu_{ji}-1) = 2n+2p-2,$$

and hence the highest order of a special virtual series is

$$k+2p-2 = 2n+4p-4.$$

Since, by hypothesis, $m > 2p+2$, the above characteristic series is *non-special.* Using the formula (13), which gives the dimension of a virtually complete non-special series, we find

$$\rho = (2m+2n+2p-2)-p-k-\omega = 2m-p+\omega.$$

It follows, that the characteristic series of K on C_0 is virtually complete.

With regard to this result, we want to make the following remark. The noted theorem, announced by F. Enriques in 1904,* that the characteristic series of a complete continuous system of algebraic curves is complete, is absolutely true if the curves of the system have ordinary double points only. However, it was pointed out by F. Severi † that if the curves of the system possess higher singularities, the characteristic series may not be complete.

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* This is a particular case of a more general theorem, due to Enriques, "Sulla proprietà caratteristica delle superficie irregolari," *R. Accademia delle Scienze di Bologna*, Vol. 1 (1904), p. 5.

† "Sulla teoria degl'integrali semplici di 1ª specie, appartenenti ad una superficie algebrica," *Rendiconti della R. Accademia dei Lincei* (5), Vol. 30 (1921).

For instance, in the case recently considered by B. Serge * of a continuous system of curves possessing a variable tacnode, the characteristic series is not complete. However, the above result regarding the characteristic series of the system K suggests the possibility of restoring, at least formally, the validity of Enriques' theorem, by considering the *characteristic series* as a *virtual* series. For instance, in the above example considered by Segre, the characteristic series on a generic curve of the system is easily seen to possess the two points on the branches of the tacnode as a neutral pair, and as such the characteristic series is *virtually complete.* In the quoted paper Segre also considers a system of curves having a variable ordinary r-fold point P. Here, if we denote by $P_1, P_2, \cdots, P_n$ the n points of the curve on the branches passing through P, it is easily seen that *the residual series* of the characteristic series with respect to any of the points P_i possesses the remaining $n-1$ points as a neutral set, and as such this residual series is virtually complete. We do not intend here to pursue the investigation of this interesting question. We are satisfied with having called attention to an interpretation of Enriques' theorem, which may be useful for the study of continuous systems of curves.

8. We now come back to the characteristic series of the system H on C_0. We know already (see section 5) that the order and the dimension of this series are $2n$ and $\omega - 3p + i + 2$, and that the sets (4) of sec. 3 are neutral sets of the series. If i' is the index of speciality of the characteristic series, then the dimension of the virtually complete series $\| g_{2n} \|$ is [see (14), sec. 6]

$$2n - p - k + \omega + i' = \omega - 3p + 2 + i'.$$

Hence $i' \geqq i$, where the sign $=$ holds, if the above characteristic series is virtually complete. Now, it can be proved that in fact i' *is* equal to i, by repeating almost literally the reasoning of Severi (see quoted paper in sec. 3). It is only necessary to observe that the theorem that the residual series of a complete series with respect to a fixed set of points is complete, *holds also in the case of virtually complete series.* It follows, as in the case considered by Severi, that the σ points of contact of the tangents λ to C_0 through B impose $\sigma - i'$ conditions only to the curves of K infinitely near to C_0. Therefore etc.

To complete the proof of the theorem announced in sec. 3, we have only to show that the characteristic series of the system H in C_0 contains each set $\Gamma_1 + \Gamma_2$ made up of two sets of the g_n^1 cut out on C_0 by the lines of the pencil A. To show this, let A and B be taken at infinity on the x and y axes

* "Sui sistemi continui di curve piane con tacnodo, *Rendiconti della R. Accademia dei Lincei* (6), Vol. 9 (1929).

respectively, and let $f(x, y) = 0$ be the equation of C_0. Among the curves of K infinitely near to C_0 we find evidently the transforms of C_0

$$f[(x + a(t), \quad y + b(t)] = 0,$$

by an infinitesimal translation. The corresponding characteristic sets are cut out by the polar curves of C_0,

$$a_1(\partial f/\partial x) + b_1(\partial f/\partial y) = 0,$$

with respect to the points on the line at infinity. In particular, the characteristic set cut out on C_0 by the polar curve $\partial f/\partial y = 0$ is made up of the $\sigma = 2m + 2p - 2$ points of contact of the tangents λ and of the n points $A_1, A_2, \cdots, A_n$, *each counted twice,* which lie on the n linear branches of C_0 through A. The points $A_1, A_2, \cdots, A_n$ form evidently a set of the g_n^1. Hence, the characteristic series $\| g_{2n} \|$ contains the set $[A_1^2, A_2^2, \cdots, A_n^2]$. Since the ∞^2 sets of $2n$ points made up of 2 arbitrary sets of the g_n^1 form a linear series g_{2n}^2 possessing the same ω neutral sets as the virtually complete series $\| g_{2n} \|$, and since both series contain a common set $[A_1^2, A_2^2, \cdots, A_n^2]$, we deduce that the above g_{2n}^2 is contained in the series $\| g_{2n} \|$.

9. In order to apply the theorem of sec. 3 to the group problem announced in sec. 1, it would now be necessary to investigate the virtually complete series $\| g_{2n} \|$, with special reference to the question of its index of speciality. In the case $\omega \geqq 3p$ (assuming always that G is a primitive group), the aim of this investigation would be to prove, if possible, that the above series is non-special in general, i. e. when the ω branch points are arbitrary. In the case $\omega < 3p$, the above series is certainly special, since it contains the g_{2n}^2 obtained by combining the sets of the g_n^1 two at a time. Hence, if i is the index of speciality of the complete series $\| g_{2n} \|$, then

$$\omega - 3p + 2 + i \geqq 2,$$

or

$$i \geqq 3p - \omega.$$

If the sign $=$ holds, then the functions y of Σ depend on $3p - 3 - (3p - \omega) = \omega - 3$ moduli. Otherwise the number of moduli is less than $\omega - 3$.

The treatment of the above outlined questions involves the study of certain continuous systems of curves possessing singularities other than ordinary double points, and may form the subject of another paper. Here, instead, we wish to make some critical remarks concerning a statement in the paper by Severi quoted above, and we also wish to call attention to a slight error in one of the theorems, stated by B. Segre in his otherwise most

interesting paper,* in which he applies Severi's results to the problem of the moduli of the polygonal curves of genus p.

10. In section 5 of his paper Severi states the following theorem: *On a curve of genus p ($p > 1$) of general moduli the two-fold of any linear series, ∞^1 at least, of order ν (ν arbitrary) is a non-special series.* In stating this theorem Severi remarks that it is an immediate consequence of his general theorem on the index of speciality of the two-fold of a series g_ν^1 on a curve of genus p (see this paper, sec. 3). However, it seems to us that, rigorously speaking, this general theorem allows us only to affirm that the two-fold of a *generic* linear series on a curve of genus p of general moduli is non-special. In fact, if the curves of the system H, considered by Severi, are of general moduli, then it is perfectly correct that the two-fold of the g_ν^1 on a *generic* curve C of H is a non-special series. But for particular curves of H this series may well become special, and it is not evident at all that the curves of H, for which this happens, are necessarily of *special moduli*. Nothing prevents us from thinking that these particular curves of the system H correspond to *particular* g_ν^1's on curves of genus p of general moduli. The theorem that the two-fold of any series, at least ∞^1, on a curve of genus p of general moduli is non-special, if true (and most probably it is true), must yet be proved.

11. In his quoted paper B. Serge proves that the ν-gonal curves of genus p ($\nu < p/2 + 1$), i. e. the curves of genus p possessing a g_ν^1, depend on $\omega - 3 = 2\nu + 2p - 5$ moduli. In view of Severi's theorem, it all amounts to proving the theorem that the index of speciality i of the two-fold of a generic g_ν^1 on a general ν-gonal curve of genus p is

$$i = p - 2\nu + 2.$$

Segre's proof of this theorem is somewhat lengthy. We give here a simple proof of this theorem, using the system H considered by Severi. The curves C of H are of order $n + \nu$, of genus p, and possess one fixed ν-fold point P, another fixed n-fold point Q, and $d = (\nu - 1)(n - 1) - p$ variable ordinary double points.

The adjoint curves of order $n + \nu - 3$ of C degenerate into the fixed line PQ and into curves C_1 of order $n + \nu - 4$ with a $(\nu - 2)$-fold point at P and a $(n - 2)$-fold point at Q and passing simply through the double points of C. Let t_1 and t_2 be two generic lines through Q. *We show that there*

* B. Segre, "Sui moduli delle curve poligonali, e sopra un complemento al teorema di esistenza di Riemann," *Mathematische Annalen*, Vol. 100 (1928).

exist exactly $\infty^{p-2\nu+1}$ *curves* C_1 *passing through the* 2ν *intersections (outside of* Q*) of the lines* t_1, t_2 *with* C, *and that hence the index of speciality* i *of the two-fold of the* g_ν^1, *cut out by the lines of the pencil* Q *on the generic curve* C *of* H, *is* $p-2\nu+2$.

Let us denote by K the system of the curves C_1, and let δ be the dimension of K. The curves C_1 evidently degenerate into the two lines t_1 and t_2 and into residual curves C_2 of order $n+\nu-6$, which must possess a $(\nu-2)$-fold point at P, an $(n-4)$-fold point at Q and must pass simply through the d double points of C. The system H contains degenerate curves made up of ν arbitrary lines $a_1, a_2, \cdots, a_\nu$, passing through P, and n arbitrary lines $b_1, b_2, \cdots, b_n$ passing through Q. Let D_0 be one of these degenerate curves. As the curve C approaches D_0, the d double points of C tend to d *assigned* double points of D_0 (while the remaining $n+\nu+p-1$ double points of D_0 have to be considered as *virtually non-existent*), and the curves C_2 will tend to curves Γ_1 of order $n+\nu-6$, behaving in the same way at P and Q as the curves C_2, and passing simply through the d assigned double points of D_0. The number of arbitrary parameters, on which the limit curves Γ_1 depend, evidently cannot be less than the number δ of arbitrary parameters, on which the curves C_2 depend. Hence if we call Σ_1 the system of the curves Γ_1, and if we denote by δ_1 the dimension of Σ_1, we have

$$\delta_1 \geqq \delta. \tag{16}$$

We proceed to evaluate δ_1. We first agree to choose the $n+\nu+p-1$ virtually non-existent double points of D_0 as follows: Let us consider the n points in which the line a_1 is met by the lines $b_1, b_2, \cdots, b_n$, and the $3(\nu-1)$ points in which the lines b_1, b_2, b_3 are met by the lines $a_2, a_3, \cdots, a_\nu$. We have thus considered in all $n+3(\nu-1)$ double points of D_0. The remaining

$$n\nu-n-3(\nu-1)=(n-3)(\nu-1)$$

double points of D_0 are the double points of the curve $D_0'=a_2+a_3+\cdots+a_\nu+b_4+b_5+\cdots+b_n$. Since by hypothesis $\nu<p/2+1$, we have

$$n+\nu+p-1>n+3(\nu-1),$$

and hence we may assume the $n+3(\nu-1)$ double points of D_0 considered above to be virtually non-existent, and choose the remaining

$$(n+\nu+p-1)-(n+3\nu-3)=p-2\nu+2>0$$

virtually non-existent double points from the double points of the curve D_0'. With this choice of the $n+\nu+p-1$ virtually non-existent double points

of D_0, *the d assigned double points of* D_0 *are double points of* D_0'. For the dimension δ_1 of the system Σ_1 we evidently have

$$\delta_1 \geqq \binom{n+\nu-4}{2} - \binom{\nu-1}{2} - \binom{n-3}{2} - d - 1 = p - 2\nu + 1,$$

where the sign $=$ holds, if the d assigned double points of D_0 impose d *independent conditions* on the curves Γ of order $n+\nu-6$ having a $(\nu-2)$-fold point at P and an $(n-4)$-fold point at Q. To prove that these conditions are indeed independent, we have only to observe that the $(\nu-1)(n-3)$ double points of D_0' impose as many independent conditions on the curves Γ, or, what is the same thing, that the dimension of the system of those curves Γ, which pass through all the double points of D_0', is

$$\binom{n+\nu-4}{2} - \binom{\nu-1}{2} - \binom{n-3}{2} - (\nu-1)(n-3) - 1 = -1,$$

i. e. that such curves Γ do not exist. In fact, a curve Γ, which passes through all the double points of D_0' meets each of the lines $a_2, a_3, \cdots, a_\nu, b_4, \cdots, b_n$ in $(\nu-2)+(n-3)=(\nu-1)+(n-4)=n+\nu-5$ points, and hence necessarily contains each of these lines. But this is impossible, since the order $n+\nu-6$ of Γ is less than the order of the curve $a_2+a_3\cdots+a_\nu+b_4+\cdots+b_n$.

Hence, the $(\nu-1)(n-3)$ double points of D_0' impose independent conditions on the curves Γ, and since the d assigned double points of D_0 are among the double points of D_0', it follows that they also impose independent conditions on the curves Γ. Hence $\delta_1 = p - 2\nu + 1$.

From (16) we deduce that

$$\delta \leqq p - 2\nu + 1,$$

and, since obviously δ cannot be less than $p-2\nu+1$, it follows that

$$\delta = p - 2\nu + 1, \qquad \text{q. e. d.}$$

12. Having thus established the theorem, that the index of speciality of the two-fold of a series g_ν^1 $(\nu < p/2+1)$ on a curve of genus p is *in general* $i = p-2\nu+2$, one may ask on how many moduli do the particular ν-gonal curves, for which the above index of speciality is greater than $p-2\nu+2$, depend. In his quoted paper (p. 545) Segre states that the ν-gonal curves, for which

$$i = p - 2\nu + 2 + \sigma, \qquad (\sigma \geqq 1)$$

depend on $2p+2\nu-5-\sigma$ moduli. We cannot agree with this statement, since we have found that, already in the case $\sigma = 1$, the number of moduli

is less than that given by Segre. It may seem that the above statement is an immediate consequence of Severi's theorem (number of moduli $= 3p - 3 - i$). But it must be emphasized that this theorem says only that, *if i is the index of speciality of the two-fold of a g_ν^1 on the general ν-gonal curve of genus p, then the ν-gonal curves depend on* $3p - 3 - i$ moduli. Thus, having found that in general $i = p - 2\nu + 2$, we have deduced that the ν-gonal curves of genus p depend on $2\nu + 2p - 5$ moduli. But Severi's theorem does not say that the *particular* ν-gonal curves, for which the above index of speciality has a given value $i > 2\nu - 2p + 2$, depend on $3p - 3 - i$ moduli.* To give an illustration, let us evaluate the number of the moduli in the case $\sigma = 1$.

LEMMA. *If $(\nu - 1)^2 \geqq p$, there exists an irreducible continuous system $\infty^{4\nu+p-1}$ of plane irreducible curves C of order 2ν and genus p, which possess two fixed infinitely near ν-fold points O, O_1 and $(\nu - 1)^2 - p$ variable ordinary double points.*

We consider the plane curves D of order 2ν possessing a ν-fold point at a fixed point O, and we suppose that O is the origin of ν linear branches having a common fixed tangent t. Under this hypothesis the curves D possess a ν-fold point O_1, infinitely near to O in the direction of the fixed tangent t. The differential condition that a curve of order 2ν should possess a ν-fold point at a fixed point O and a ν-fold point at a point O_1 infinitely near to O on a fixed line through O is expressed by $\nu(\nu + 1)$ linear relations among the coefficients of the equation of the curve, and it is easily seen that these relations are independent.† Hence the curves D form a linear system Σ of dimension

$$\nu(2\nu + 3) - \nu(\nu + 1) = \nu(\nu + 2).$$

The curves of Σ are irreducible, since Σ contains curves D which degenerate into ν arbitrary conics passing through O and touching the line t at O, which shows that the curves of Σ cannot be composed of the curves of a pencil.

Let D_0 be one of the above reducible curves of D, which degenerates into ν conics $c_1, c_2, \cdots, c_\nu$. *We say that the $\nu(\nu - 1)$ double points of D_0 (distinct from O) impose as many independent conditions on the curves D.* In fact let ρ be the dimension of the system Σ' of the curves D passing through the double points of D_0. Then

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* Here the question arises whether the family of the said particular ν-gonal curves is irreducible. See Remark 1, p. 171.

† See, F. Enriques and O. Chisini, "Teoria geometrica delle equazioni e delle funzioni algebriche," *Zanichelli, Bologna*, Vol. 2, p. 471.

$$\rho \geqq \nu(\nu+2)-\nu(\nu-1)=3\nu, \tag{17}$$

where the sign $=$ holds, if the above conditions are independent.

The curves of Σ' meet the conic c_1 in

$$2\nu+2(\nu-1)=4\nu-2$$

fixed points, and hence the dimension of the system Σ'' of those curves Σ', which degenerate into the conic c_1 and into a residual curve of order $2(\nu-1)$, is $\geqq \rho-3$. The curves of Σ'' have a $(\nu-1)$-fold point at O and a $(\nu-1)$-fold point at O_1, and pass through the double points of the reducible curve $c_2+c_3+\cdots c_\nu$. Accordingly they meet the conic c_2 in

$$2(\nu-1)+2(\nu-2)=4(\nu-1)-2$$

fixed points, and consequently the dimension of the system Σ''' of those curves of Σ'', which degenerate into the conic c_2 and into a curve of order $2(\nu-2)$, is $\geqq \rho-6$. This procedure leads after ν steps to the conclusion that the curves of Σ' which degenerate into the conics $c_1, c_2, \cdots, c_\nu$ form a system of dimension $\geqq \rho-3\nu$. But this last system is of dimension 0, since it contains only one curve, viz., the curve $c_1+c_2+\cdots+c_\nu$. Hence,

$$\rho \leqq 3\nu. \tag{18}$$

From (17) and (18) we deduce that

$$\rho = 3\nu.$$

Hence the $\nu(\nu-1)$ double points of the degenerate curve D_0 impose $\nu(\nu-1)$ linearly independent conditions on the curves of the system Σ.

We now consider *as virtually non-existent* $\nu+p-1$ of the $\nu(\nu-1)$ double points of D_0, so that D_0 becomes a connected curve with two infinitely near assigned ν-fold point at O, O_1 and other $\nu(\nu-1)-(\nu+p-1)=(\nu-1)^2-p=d$ assigned double points. The virtual genus of D_0 will be then

$$\binom{2\nu-1}{2}-2\binom{\nu}{2}-d=p.$$

In order that D_0 should become a connected curve, we may consider, for instance, as virtually non-existent one of the two intersections (distinct from O) of c_1 with each of the conics $c_2, c_3, \cdots, c_\nu$ and other p points chosen arbitrarily from the remaining $(\nu-1)^2$ double points of D_0.* In the linear

* For the concepts which we use here, and also for the method of "analytical regions" ("falde analitiche," "analytische Mantel") see Severi, quoted treatise, *Anhang F.* See also quoted treatise by *Enriques-Chisini*, Vol. 3, Chap. 3, § 33.

space $S_{\nu(\nu+2)}$ of dimension $\nu(\nu+2)$, the points of which represent the curves of the system Σ, the curves of Σ possessing one double point in the neighborhood of one of the assigned double points of D_0 are represented by a linear analytical region Φ_i $(i=1,2,\cdots d)$ having its origin at D_0. Since the assigned d double points of D_0 impose linearly independent conditions on the curves of Σ, the d tangent spaces of the regions Φ_i at D_0 are linearly independent. We deduce that the Φ_i have in common an analytical region Φ of dimension

$$\nu(\nu+2)-d=4\nu+p-1.$$

The curves of Φ possess $d=(\nu-1)^2-p$ double points in the respective neighborhoods of the assigned double points of D_0, *and hence* D_0 *belongs to a complete continuous system* H *of curves* C *of order* 2ν, *possessing the two infinitely near* ν*-fold points* O, O_1 *and* $(\nu-1)^2-p$ ordinary double points.

That the generic curve C of H possesses only ordinary double points (distinct from O and O_1), follows from the fact that this holds for the particular curve D_0 of H. Moreover the generic curve C of H does not have more than $(\nu-1)^2-p$ double points, because, if it had $d+1$ double points, then, as C approaches D_0, the double points of C would tend to $d+1$ double points of D_0. Since these $d+1$ double points of D_0 would also impose independent conditions on the curves of Σ, it would follow that the dimension of H is $4\nu+p-2$ and not $4\nu+p-1$ as above. Hence, the generic curve of H has exactly $(\nu-1)^2-p$ double points (distinct from O and O_1) and therefore is of genus p.

It remains to prove that the curves of H *are irreducible.* This follows from the fact that on the degenerate curve D_0 any two points can be joined by a path which does not meet any of the $(\nu-1)^2-p$ assigned double points, and that therefore also on the generic curve C of H it is possible to connect any two points by a continuous path without meeting any of its double points. The Lemma is thus proved.

On a generic curve C of the system H the lines of the pencil O cut out a linear involution g_ν^1, and hence C is a ν-gonal curve of genus p. The ∞^3 conics C passing through O and touching the fixed tangent t at O (and, in particular, the ∞^2 lines of the plane) cut out on C sets of 2ν points belonging to the two-fold of the series g_ν^1. We prove that, *if* $\nu \leqq p/2+1$, *the series* $g_{2\nu}^3$ *cut out by the above conics is complete, or that the index of speciality* i *of the two-fold of the* g_ν^1 *is* $p-2\nu+3$.

To find i, we have to find the number of linearly independent adjoint curves of order $2\nu-3$ of C, which pass through the 2ν intersections (distinct from O) of one of the above conics c with the curve C. The adjoint curves

of order $2\nu-3$ of C possess two infinitely near $(\nu-1)$-fold points at O, O_1, and hence degenerate into the fixed line t and into residual curves of order $2\nu-4$ possessing two infinitely near $(\nu-2)$-fold points at O, O_1, and passing simply through the remaining $(\nu-1)^2-p$ double points of C. The adjoint curves of order $2\nu-3$ of C, which pass through the above 2ν intersections of the conic c with the curve C, degenerate into the fixed line t, into the conic c and into residual curves C_1 of order $2\nu-6$ possessing two infinitely near $(\nu-3)$-fold points at O, O_1 and passing simply through the remaining $(\nu-1)^2-p$ double points of C. Since the dimension of the system of the curves C_1 is, by hypothesis, $i-1$, we have

$$(19) \qquad i \geqq \binom{2\nu-4}{2} - 2\binom{\nu-2}{2} - (\nu-1)^2 + p = p - 2\nu + 3.$$

Now, let the curve C tend to the degenerate curve $D_0 = c_1 + c_2 + \cdots + c_\nu$. The curves C_1 will become in the limit, curves Γ of order $2\nu-6$, behaving in the same way as the curves C_1 at the points O, O_1 and passing simply through the $(\nu-1)^2-p$ assigned double points of D_0. Hence, denoting by r the dimension of the complete linear system defined by the base points of the curves Γ, we will have

$$(20) \qquad r \geqq i-1.$$

In order to evaluate r, we first agree to choose the $(\nu-1)^2-p$ assigned double points of D_0 as follows: We consider the following $3(\nu-1)$ double points of D_0: (a) The $2(\nu-1)$ points at which the conic c_1 is met by the conics $c_2, c_3, \cdots, c_\nu$; (b) the two intersections of the conics c_2, c_3; (c) one of the two points at which c_2 is met by each conic $c_4, \cdots, c_\nu$. If the above $3(\nu-1)$ double points of D_0 are considered as virtually non-existent, D_0 becomes a connected curve. Since, by hypothesis, $\nu \leqq p/2+1$, and hence $\nu+p-1 \geqq 3(\nu-1)$, we may assume the considered $3(\nu-1)$ double points of D_0 to be virtually non-existent, and choose arbitrarily the remaining

$$\nu+p-1-3(\nu-1) = p-2\nu+2 \geqq 0$$

virtually non-existent double points of D_0 from the remaining

$$\nu(\nu-1)-3(\nu-1) = (\nu-1)(\nu-3)$$

double points of D_0. Let us call A this last set of $(\nu-1)(\nu-3)$ double points of D_0.

The $(\nu-1)(\nu-3)$ *points* A *impose as many independent conditions on the curves* E *of order* $2\nu-6$ *possessing two* $(\nu-3)$-fold points at O, O_1.

In fact, let r' be the dimension of the system K of those curves E which pass through the points A. Then

$$r' \geqq \binom{2\nu-4}{2} - 2\binom{\nu-2}{2} - (\nu-1)(\nu-3) - 1 = 0,$$

where $r' = 0$, if and only if the points A impose $(\nu-1)(\nu-3)$ independent conditions on the curves E. Now let us observe that each of the conics $c_4, c_5, \cdots, c_\nu$ contains $2\nu-5$ points A and that consequently each curve E of K must contain each of the above conics, since each curve E of K has already, outside of the points A, $2\nu-6$ coincident intersections with E at O. It follows that the system K contains only one curve, viz. the degenerate curve $c_4 + c_5 + \cdots + c_\nu$, and hence $r' = 0$.

Since the $(\nu-1)^2 - p$ assigned double points of D_0 are all chosen from the points A, they also impose independent conditions on the curves E. It follows that

$$r = \binom{2\nu-4}{2} - 2\binom{\nu-2}{2} - (\nu-1)^2 + p - 1 = p - 2\nu + 2,$$

and hence, from (20),

$$i \leqq p - 2\nu + 3. \tag{21}$$

Comparing (21) with (19) we deduce that

$$i = p - 2\nu + 3, \qquad \text{q. e. d.}$$

Let C_0 be a generic curve of H, and let us consider the $\sigma = 2\nu + 2p - 2$ tangents λ to C_0 through O. We consider the system of curves C of H, which touch the tangents λ, and we denote by H_0 that irreducible part of this system which contains C_0. It is easily seen that the sets of characteristic series of H in C_0 are sets of the two-fold of the series g_ν^1. On the other hand, by a theorem, due to Severi, on continuous systems of curves constrained to touch a fixed set of lines of a pencil, which we have used at an earlier stage of this paper (see sec. 8), and which applies without any modification to the system H_0, we deduce that the characteristic series of H_0 on C_0 is *the complete series* $g_{2\nu}^3$.

It follows that the σ tangents λ depend only on $4\nu + p - 1 - 4 = 4\nu + p - 5$ arbitrary parameters, and that hence the curves C of H depend *at most* on $4\nu + p - 8$ moduli. It will be noted that if $\nu < p/2 + 1$, the derived upper limit $4\nu + p - 8$ of moduli is certainly less than the number $2\nu + 2p - 6$ evaluated according to Segre's formula. *We have thus established the existence of a complete irreducible system H of ν-gonal curves of*

genus p, for which the two-fold of the series g_ν^1 is a special series of index of speciality $p-2\nu+3$, and which depend at most on $4\nu+p-8$ moduli. We observe that, in order to prove that the curves of H depend exactly on $4\nu+p-8$ moduli, it is necessary to prove that the generic curve of H does not possess infinite linear involutions g_ν^1.

Remark 1. If a ν-gonal curve of genus p contains a g_ν^1 such that the two-fold of the g_ν^1 is a $g_{2\nu}^3$ (at least), the curve can be transformed into a plane curve of order 2ν possessing two fixed infinitely near ν-fold points and other singularities equivalent to $(\nu-1)^2-p$ double points, *provided the $g_{2\nu}^3$ is not composed of an involution.* In order to prove that the ν-gonal curves for which the above $g_{2\nu}^3$ is not composed of an involution form an irreducible system, it would be necessary to prove that the plane curves of order 2ν and of genus $p \leqq (\nu-1)^2$, possessing two fixed infinitely near ν-fold points, form a unique irreducible system, which then coincides with the system H considered above.

Under the hypothesis that the $g_{2\nu}^3$ is composed of an involution, it can be easily verified that the corresponding ν-gonal curves may form a complete irreducible system of dimension $> 4\nu+p-8$ and hence distinct from H.

Remark 2. We suggest here a formula for the number of moduli of the ν-gonal curves of genus p possessing a g_ν^1, such that the two-fold series of the g_ν^1 is of index of speciality $i = p-2\nu+2+\sigma$ (always under the hypothesis that the corresponding complete series $g_{2\nu}^{\sigma+2}$ is not composed of an involution). This formula is as follows:

$$\begin{aligned}\text{number of moduli} &= 2\nu+2p-5-\sigma(p-2\nu+2+\sigma)\\ &= 2\nu+2p-5-\sigma i.\end{aligned}$$

The above formula gives, for $\sigma=1$, the derived number $4\nu+p-8$ above, and for any $\sigma>0$ is suggested by the following hyperspacial considerations, relative to the canonical curve of genus p in a space S_{p-1}. Assuming that the canonical curve C is a ν-gonal curve, each of the ∞^1 sets Γ_ν of the g_ν^1 belongs to a space $S_{\nu-2}$. The index of speciality of the two-fold of the series g_ν^1 being $p-2\nu+2+\sigma$, any two of the spaces $S_{\nu-2}$ meet in a space $S_{\sigma-1}$. We know that if C is a generic ν-gonal curve, then $\sigma=0$, i. e. the spaces $S_{\nu-2}$ do not meet one another. It is also easily seen that the number of conditions, which express the fact that two spaces $S_{\nu-2}$, given in a S_{p-1}, meet in a $S_{\sigma-1}$, is $\sigma(p-2\nu+2+\sigma)=\sigma i$. The above formula is derived under the hypothesis that to the above number of incidence conditions corresponds an equal number of independent conditions on the moduli of the curve C.

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On Self-Adjoint Ordinary Differential Equations of the Fourth Order.*

By William M. Whyburn †

The work of Sturm, Bôcher, and others on real second order differential equations forms an important chapter in mathematical analysis. Birkhoff ‡ obtained many important results for third order differential systems and introduced a new and valuable method of proving comparison theorems for such systems. C. N. Reynolds § used Birkhoff's methods to study equations of the n^{th} order. Hille ‖ studied the second order equation in the complex domain and discovered many important facts concerning the distribution of the zeros of the solution in the complex plane. Davidoglou ¶ gave a detailed treatment of a special fourth order self-adjoint equation that arose in connection with a study of the lateral vibrations of a rod. If certain factors, such as the rotational inertias, are taken into account in the problem of the vibrating rod, the equation that must be studied is the general self-adjoint fourth order equation rather than the special equation treated by Davidoglou.** H. T. Davis †† has recently considered this equation.

It is the intention of the present paper to study the general self-adjoint equation of the fourth order

$$\frac{d^2}{dx^2}\left[K(x)\frac{d^2u}{dx^2}\right]+\frac{d}{dx}\left[L(x)\frac{du}{dx}\right]+G(x)u=0 \tag{1}$$

and the second order equation

$$\frac{d^2u}{dx^2}+[p_1(x)+ip_2(x)]\frac{du}{dx}+[q_1(x)+iq_2(x)]u=0, \tag{2}$$

where K, L, G, p_1, p_2, q_1, q_2 are real functions of the real variable x and i is

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† National Research Fellow in Mathematics.

‡ *Annals of Mathematics*, Ser. 2, Vol. 12 (1910-11), pp. 103-127.

§ *Transactions of American Mathematical Society*, Vol. 22 (1921), pp. 220-229.

‖ *Transactions of American Mathematical Society*, Vol. 23 (1922), pp. 350-385.

¶ *Annales de l'Ecole Normale Superieure* (3), Vol. 22, pp. 539 ff.

** Cf. Lord Rayleigh: *Theory of Sound*, London, 1894, Vol. I, Chapter VIII.

†† *American Journal of Mathematics*, Vol. 47 (1925), pp. 101-120.

the imaginary symbol. This study is accomplished with the aid of a pair of equations of the second order.* The paper treats system (3) for the cases $qr < 0$ on the given interval and $qr > 0$ on that interval. Extensive use is made of the associated second order equation (6) and in many cases the properties of solutions of system (3) are derived from, or related to, similar properties of solutions of (6). Use is made of a combination $V(x)$ of y and z and by means of the properties of this function and transformations involving solutions of equation (6) separation theorems are established for the zeros of $y(x)$ and $z(x)$ in the case $qr < 0$. In the presence of additional restrictions on p, q, and r, it is shown that for every solution of (3) $y(x)$ and $z(x)$ oscillate infinitely often on the real axis and the integral curve in the yz-plane is shown to be essentially *spiriliform*. Related and equivalent integral equations are used in Part III to develop comparison theorems on "regular" intervals for system (3) in the case $qr > 0$. The paper leaves the cases that arise when qr changes sign on the interval wholly untouched. The case $qr < 0$ is emphasized because it is the one which arises in a simultaneous treatment of equations (1) and (2). This point is brought out more clearly near the end of Part I.

I. TRANSFORMATION OF THE EQUATIONS.

We follow the well established custom of stating that $f(x)$ is of class $C^{(j)}$ on X: $a \leqq x \leqq b$ provided $f(x)$ together with its first j derivatives is continuous on X.

THEOREM I. *If, in equation* (1), K, L'', G *are continuous and* $K \neq 0$ *on* X, *equation* (1) *can be written in the form*

$$(3) \qquad \begin{aligned} &(a)\ y'' + p(x)y = q(x)z \\ &(b)\ z'' + p(x)z = r(x)y, \end{aligned}$$

where p, q, *and* r *are continuous on* X *and furthermore,* $q(x) \neq 0$ *and* p/q *is of class* C'' *on* X. *Conversely, if* p, q, *and* r *are continuous on* x *and* $q \neq 0$, p/q *is of class* C'' *on* X, *then system* (3) *is equivalent to equation* (1) *where* $K \neq 0$ *and* K, L'', *and* G *are continuous on* X.

Proof. Let $q = 1/K$, $p = L/2K$, $r = L''/2 + L^2/4K - G$. Solving these equations for K, L, and G *yields*

$$K = 1/q,\ L = 2p/q,\ G = (p/q)'' + p^2/q - r$$

* The author is indebted to Prof. G. D. Birkhoff for many valuable suggestions that have furthered the writing of this paper.

and it becomes evident that p, q, and r have the desired properties of continuity, etc.

Substitution into equation (1) yields

$$[u''/q]'' + 2[pu'/q]' + u(p/q)'' + up^2/q - ru = 0$$
$$[u''/q + pu/q]'' + p[u''/q + pu/q] = ru.$$

Let $u = y$ and $z = u''/q + pu/q$. Simplification after this substitution yields system (3).

To prove the converse, we divide equation (3) (a) by q and compute z'' from the resulting equation. Upon substituting this value into equation (3) (b) and simplifying, one obtains equation (1).

In equation (2) let p_1 and p_2 be of class C' on X while q_1 and q_2 are continuous on that interval. Let $u = W \exp\left[-\tfrac{1}{2}\int_a^x (p_1 + ip_2)\,dt\right]$, $Q = p + iq$, where $p = q_1 - p_1^2/4 + p_2^2/4 - p_1'/2$ and $q = q_2 - p_1p_2/2 - p_2'/2$. Equation (2) becomes

$$W'' + Q(x)W = 0, \tag{4}$$

where Q is continuous on X.

The ordinary existence theorem * insures the existence of a unique solution † of (4) that is of class C'' on X and such that if $x = c$ is any point of X and α, β, γ, δ, are arbitrarily assigned real constants, this solution takes on the initial values $W(c) = \alpha + \beta i$, $W'(c) = \gamma + \delta i$. A similar theorem applied to the system

$$y'' + p(x) = q(x)z, \qquad z'' + p(x)z = -q(x)y \tag{5}$$

shows the existence of a unique pair of functions of class C'' on X that satisfies (5) on X and takes on the initial values $y(c) = \alpha$, $y'(c) = \gamma$, $z(c) = \beta$, $z'(c) = \delta$. Let $W = y + iz$. Direct substitution shows that W is a solution of (4) that assumes the given initial values. It follows from the uniqueness of such solutions that (4) with its initial conditions is equivalent to (5) with the corresponding initial conditions. It also follows that any solution of (4) that is of class C'' on X can be written in the form $y + iz$, where y and z are real functions of the real variable x.

Thus we are lead to a consideration of the pair of real equations (3). Since the transformation of Theorem I carries the non-singular equation (1)

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* Cf. Bôcher, *Leçons sur les Méthodes de Sturm*, Paris (1917), page 1.

† Throughout the paper we consider only those solutions of a differential equation that have continuous derivatives of as high order as the order of the equation.

into a system (3) where $q(x)$ does not vanish on X while equation (2) yields a system (3) with $r(x) = -q(x)$, we are led to a study of (3) where $q(x)r(x) < 0$ in order to obtain results that carry over to both equation (1) and equation (2).

Let x be regarded as a parameter, then the pair of functions $y(x)$, $z(x)$ given by system (3) describes a curve in the yz-plane as x ranges over the set of values X. This curve is called the *integral* curve of (3) in the yz-plane and its shape is evidently dependent upon the initial values as well as the coefficients of the system. We propose to study this curve. We use the notation (R, θ) for the polar co-ordinates of a point on this curve and hence $R(x)^2 = y(x)^2 + z(x)^2$, $\theta(x) = \tan^{-1}(z/y)$.

The following dynamical interpretation may be given for system (3) following a similar one given by Hille [1] for system (5). If x is regarded as the time, then the integral curve of (3) in the yz-plane is the path of motion of a particle that is acted upon by a force of magnitude $|pR|$ along the radius vector, a force of magnitude $|qR|$ perpendicular to the radius vector, and a force of magnitude $|(r-q)y|$ parallel to the z-axis. Other distributions of the forces may be used. In the light of this interpretation, the hypotheses and conclusions of later theorems in the paper have a definite physical significance.

II. SYSTEM (3) WITH $q(x)r(x) < 0$.

Throughout this part we suppose that p, q, and r are continuous on the interval under consideration and in addition that $qr < 0$ on this interval.† Let

$$V(x) = V[y(x), z(x)] = y'(x)z(x) - z'(x)y(x).$$

If equation (3) (b) is multiplied by $y(x)$ and this result subtracted from the result of multiplying (3) (a) by $z(x)$, one obtains $V'(x) = q(x)z(x)^2 - r(x)y(x)$.[2] Upon integrating from $x = c$,

$$V(x) = V(c) + \int_c^x [q(t)z(t)^2 - r(t)y(t)^2]dt.$$

Since both y and z cannot vanish over an everywhere dense set of points in any subinterval of X without vanishing identically on X, we have that $V(x)$

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[2] *Loc. cit.*, page 359.

† A number of the results of the section are valid if $r(x)$ is allowed to vanish (without changing signs) over a nul set of points.

is an increasing or decreasing function of x on X according as $q(x) > 0$ on X or $q(x) < 0$ on that interval.*

THEOREM II. *The zeros of $y(x)$ and $z(x)$ separate each other on X with the possible exception of a neighborhood of one point at which $V(x) = 0$.*

Proof. Calculate $\theta'(x)$. $\theta'(x) = -V(x)/R(x)^2$, unless $R(x) = 0$. Since $V(x)$ can vanish at most once on X and since V vanishes with R, it follows that with the possible exception of one point, θ' is given by the above formula and hence can vanish or fail to exist for at most one value of x on X. Call this exceptional value c if it exists. On $a \leqq x < c$, $\theta'(x)$ is different from zero and of one sign so that θ changes continuously in one sense and the integral curve must cross the y and z axes alternately. Similarly, on $c < x \leqq b$, θ must change continuously in one sense and the zeros of y and z separate each other on this interval.

The above proof shows that under the hypotheses of this section θ can change its sense of variation at most once and the integral curve of (3) cannot pass through the origin of the yz co-ordinate system more than once.

Since $R^2\theta'$ is the rate at which area is swept out by the radius vector and the integral curve and since this quantity is equal to $-V(x)$, we have that the rate of sweeping out area increases or decreases with increase of x according as $q(x)$ is negative or positive.

THEOREM III. *If the closed interval X is of finite length, neither y nor z can vanish more than a finite number of times on X.*

Proof. Assume that y vanishes infinitely often on X and let $x = c$ be a limit point of its zeros on X. It follows from Theorem II that $x = c$ is also a limit point of the zeros of z. An application of Rolle's theorem shows that $x = c$ must be a limit point of the zeros of both $y'(x)$ and $z'(x)$. Since $y(x)$ and $z(x)$ are of class C'' on X, we have $y(c) = z(c) = y'(c) = z'(c) = 0$. It follows from the fundamental existence theorem that $y(x) \equiv z(x) \equiv 0$ on X. This, however, contradicts the fact that we are working with non-identically vanishing solutions of (3).

1°. *Case where the solutions of* (6) *oscillate on X.* Consider the second order equation

$$W'' + p(x)W = 0, \tag{6}$$

where $p(x)$ is the function that appears in system (3). It is well known †

* Throughout the paper we exclude the trivial solution $y(x) \equiv 0$, $z(x) \equiv 0$.

† Cf. Bôcher, *loc. cit.*, page 46.

that the zeros of any two linearly independent solutions of (6) separate each other on X.

THEOREM IV. *If there exists a solution of* (6) *that vanishes twice on* X, *then the product* $y(x)z(x)$ *must vanish at least once on* X *for every solution* $y(x)$, $z(x)$ *of system* (3).

Proof. Suppose that y, z is a solution of (3) such that yz does not vanish on X. Let $u(x)$ and $v(x)$ be the unique functions determined by the second order systems

$$(7)\quad [y^2u']' + qyz\,u = 0, \qquad u(a) = W(a)/y(a), \qquad u'(a) = [W/y]'_{x=a},$$
$$(8)\quad [z^2v']' + ryz\,v = 0, \qquad v(a) = W(a)/z(a), \qquad v'(a) = [W/z]'_{x=a},$$

where $W(x)$ is the solution of (6) that vanishes twice on X. The functions $u(x)y(x)$ and $v(x)z(x)$ are each solutions of (6) since

$$[uy]'' = [y^2u']'/y + uy'' = [-qz + y'']u = -pyu,$$

and similarly for vz. Furthermore, $u(a)y(a) = W(a)$, $[uy]'_{x=a} = W'(a)$, $v(a)z(a) = W(a)$, $[vz]'_{x=a} = W'(a)$. It follows from the uniqueness of $W(x)$ that

$$(9)\qquad W(x) \equiv u(x)y(x) \equiv v(x)z(x).$$

By virtue of the hypotheses on q and r we have that either $qyz < 0$ or $ryz < 0$ on X. It follows from well-known theorems * for second order equations of type (7) and (8) that either $u(x)$ or $v(x)$ cannot vanish twice on X according as $qyz < 0$ or $ryz < 0$ on that interval. Since neither y nor z vanishes on X and W vanishes twice, it follows from (9) that both u and v must vanish twice on X. This contradiction yields our theorem.

COROLLARY I. *If* (6) *has a solution that vanishes four times on* X, *then* y *and* z *each have at least one zero on* X *and the product* yz *vanishes at least three times on that interval.*†

2°. *The case* $p(x) \equiv 0$. Let $p(x)$ be identically zero and consider the resulting pair of equations. In the dynamical interpretation of Part I this means that the force along the radius vector is absent.

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* Cf. Bôcher, *loc. cit.*, page 51.

† Theorem II shows that the number of zeros of y and z on X can differ by at most two. The corollary follows from this fact and the observation that yz must vanish at least three times.

THEOREM V. *Let the coefficients of system* (3) *be continuous on the interval* X: $a \leqq x < \infty$ *and let* $p(x) \equiv 0$ *on this interval, then every solution of* (3) *such that* $V(a)q(a) \geqq 0$ *will oscillate infinitely often on* X *provided the Riemann integrals* $\int_a^b tq(t)\,dt$, $\int_a^b tr(t)\,dt$ *become infinite with* b. *That is to say, the integral curve in the* yz*-plane will circulate the origin infinitely often as* x *ranges over* X.

Proof. From the separation of the zeros of y and z, it follows that if one of these vanishes only a finite number of times the other will also. Assume that there exists a solution $y(x)$, $z(x)$ of (3) such that $V(a)q(a) \geqq 0$ and such that y and z vanish only a finite number of times on X. Let $x = d$ be chosen on X so that $d > a$ and neither y nor z vanishes on D: $d \leqq x < \infty$. For the sake of definiteness, we suppose $q > 0$ on X since the other case can be obtained from this by merely interchanging the roles of y and z in the arguments. Since $V(x)$ is an increasing function on X, it follows that $V(x) > 0$ on D and hence either: (A) $y'(d)z(d) > 0$, or (B) $y(d)z'(d) < 0$.

Consider (A). From the differential equations,

$$y'(x) = y'(d) + \int_d^x q(t)z(t)\,dt,$$

$$z'(x) = z'(d) + \int_d^x r(t)y(t)\,dt.$$

Since $y'(d)z(d) > 0$, it follows from the above formulas that $y'(x)$ is a positive increasing function or a negative decreasing function on D according as $z(x)$ is positive or negative on D. Since the same arguments treat both cases, we suppose that $z(x) > 0$ on D. We may choose a point $x = h$ on D so that $\int_d^h y'(t)\,dt > |\,y(d)\,|$ and we then have for every $x > h$, $y(x) > \int_h^x y'(t)\,dt > y'(h)(x - h)$. Now $z'(x) = z'(h) + \int_h^x r(t)y(t)\,dt > z'(h) + y'(d)\int_h^x r(t)(t - h)\,dt$. The right hand side of this inequality diverges to $-\infty$ as x increases indefinitely and hence $z(x) = z(h) + \int_h^x z'(t)\,dt$ becomes negatively infinite as x becomes infinite on D. But $z(x)$ is continuous and $z(d) > 0$. Hence $z(x)$ must vanish on D thereby contradicting our hypothesis that $z(x) \neq 0$ on D.

The arguments for (B) follow the same lines as for (A). The roles of y and z are interchanged in the proof and a contradiction is reached by showing that $y(x)$ must vanish on D.

THEOREM VI. *If the hypotheses of Theorem V are met with the exception that the integrals* $\int_a^\infty q(t)\,dt$, $\int_a^\infty r(t)\,dt$ *are required to diverge instead of the two integrals there given, then* EVERY *solution of* (3) *either oscillates infinitely often on* X *or there exists a point* $x = h$ *on* X *such that the quantity* $R^2 = y^2 + z^2$ *approaches zero steadily on* H: $h \leqq x < \infty$.

Proof. If for any x, $x = c$, on X, we have $V(c)q(c) \geqq 0$, then Theorem V applies to yield the present theorem. It remains to treat the case $V(x)q(x) < 0$ for every x on X and neither y nor z vanishes infinitely often on X. Let d be chosen so that $d > a$ and neither y nor z vanishes on D: $d \leqq x < \infty$. It follows from the differential equations that y'' and z'' cannot vanish on this interval and hence by Rolle's theorem that y' and z' can each vanish at most once on D. Let h be chosen so that none of the quantities y, z, y', z' vanish on H: $h \leqq x < \infty$. Since $V(x)q(x)$ is negative on H, it follows from the hypotheses on the integrals of q and r over X and the formula $V(x) = V(h) + \int_h^x (qz^2 - ry^2)\,dt$ that neither y^2 nor z^2 can be bounded from zero on H. Hence $yy' < 0$, $zz' < 0$ on H and $(R^2)' = 2(yy' + zz') < 0$. Furthermore, both y^2 and z^2 approach zero and hence R^2 approaches zero steadily as x becomes infinite on H.

3°. *The case* $p(x) < 0$, $p'(x) \leqq 0$ *on* X. In this section we suppose that $p'(x)$ exists on X: $a \leqq x < \infty$ and is continuous along with q and r on this interval. Furthermore, we require that $p(x) < 0$ and $p'(x) \leqq 0$ on X. In the dynamical system of Part I this requirement means that the force along the radius vector is directed away from the origin and that its magnitude can decrease only if R decreases.

THEOREM VII. *If there exists a positive constant* J *such that* $|q|/(-p)^{1/2} \geqq J$, $|r|/(-p)^{1/2} \geqq J$ on X, *then every solution of* (3) *such that* $V(a)q(a) \geqq 0$ *oscillates infinitely often on* X.

For the sake of definiteness, we suppose that $q(x)$ is positive on X since the other case is obtained from this by simply interchanging the roles of y and z in the proofs.

LEMMA I. *If* $w(x)$ *is the solution of* (6) *such that* $w(c) = 1$, $w'(c) = 0$, *where* $x = c$ *is any fixed point of* X, *then* $q(x)w(x)/w'(x) > q(x)/[-p(x)]^{1/2}$, $-r(x)w(x)/w'(x) > -r(x)/[-p(x)]^{1/2}$ on D: $d \leqq x < \infty$, *where* d *is any number that is greater than* c.

Proof of Lemma I. Since $p(x) < 0$ on X, ww' can vanish at most once on this interval. It follows from $w'(x) = -\int_c^x p(t)w(t)\,dt$ that w and w' are positive on D. Multiplying (6) by $w'(x)/(-p)$ and integrating from $t = c$ to $t = x$, we get $w(x)^2 = 1 + 2\int_c^x w''(t)w'(t)\,dt/[-p(t)]$. Integrating this by parts

$$w(x)^2 = 1 + w'(x)^2/[-p(x)] + \int_c^x (1/p)'w'(t)^2 dt.$$

Since $p' \leqq 0$ and $w' > 0$ on D, we have $w(x)^2 > w'(x)^2/[-p(x)]$ or $w/w' > 1/(-p)^{\frac{1}{2}}$ and multiplication of this inequality by q and $-r$, respectively, yields the desired inequalities.

LEMMA II. *If $x = b$ is any point of D and K and H are constants, K being positive, then*

$$\lim_{x\to\infty} \int_b^x K[\int_b^s f(t)w(t)^2 dt + H]ds/w(s)^2 = +\infty,$$

where $f(t)$ is identical with either $q(t)$ or $-r(t)$ and D, c, w have the same significance as in Lemma I.

Proof of Lemma II. Since $\lim_{x\to\infty} w(x)^2 = +\infty$, let g be so chosen that for every x on G: $g \leqq x < \infty$, $w(b)^2/w(x)^2 < \epsilon$, where $0 < \epsilon < 1$ and $g > b$. On C: $c \leqq x < \infty$, we have $w'' = -pw \geqq -p(a) > k > 0$, hence $w'(x) \geqq k(x-c)$ and $w(x) \geqq k(x-c)^2/2$. On D, $|H|/w^2 < 4|H|/k^2(x-c)^4$ and hence $\int_b^x [H/w(s)^2]ds$ converges as x becomes infinite. The proof of Lemma II will be completed if we show that

$$I = \int_b^x [\int_b^s w(t)^2 f(t)\,dt/w(s)^2]ds$$

diverges as x becomes infinite. Let x be any point of G and apply Cauchey's theorem * to the integrand of I.

$$\int_b^x w(t)^2 f(t)\,dt/w(x)^2 = f(s)w(s)[1 - w(b)^2/w(x)^2]/2w'(s),\ b < s < x,$$
$$> (1-\epsilon)q(s)w(s)/2w'(s) > (1-\epsilon)q(s)/2[-p(s)]^{\frac{1}{2}},$$

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* Cf. Pierpont, *Theory of Functions of a Real Variable*, Boston (1905), Vol. I, page 330. This theorem states that if $f(t)$ and $g(t)$ are continuous on $b \leqq t \leqq x$ while $f'(t)$ and $g'(t)$ exist on this interval and $g'(t)$ is finite and different from zero, then $f(x)/g(x) = f(b)/g(x) + [f'(s)/g'(s)]\ [1 - g(b)/g(x)]$, where $b < s < x$.

by Lemma I. Hence $I > \int_g^\infty (1-\epsilon)Jdt/2 = (1-\epsilon)(J/2)(x-g)$ and becomes infinite with x. This fact combined with the above proof that $\int_b^\infty Hds/w(s)^2$ converges as x becomes infinite on X yields Lemma II.

Proof of Theorem VII. Make the change of variable $y = wY$, $z = wZ$ on C: $c \leqq x < \infty$. Since $w(x)$ does not vanish on this interval and since $\theta = \tan^{-1}(z/y) = \tan^{-1}(Z/Y)$ is an invariant under this transformation, we will have proved our theorem if we show that $Y(x)$ and $Z(x)$ oscillate infinitely often on C. Furthermore,

$$U(x) = Y(x)Z'(x) - Z(x)Y'(x) = (yz' - zy')/w^2 = V(x)/w^2$$

and hence $U(x)$ is of the same sign as $V(x)$. The differential equations (3) become

$$(7) \qquad Y'' + 2(w'/w)Y' = qZ, \qquad Z'' + 2(w'/w)Z' = rY,$$

which when solved by the method of variation of parameters yields

$$(8) \quad \begin{aligned} &(a) \quad Y'(x) = [\int_b^\infty w(t)^2 q(t) Z(t)\,dt + w(b)^2 Y'(b)]/w(x)^2 \\ &(b) \quad Z'(x) = [\int_b^\infty w(t)^2 r(t) Y(t)\,dt + w(b)^2 Z'(b)]/w(x)^2, \end{aligned}$$

where $x = b$ is any point of C.

Now assume that Y and Z vanish only a finite number of times on C and let D: $d \leqq x < \infty$ be chosen as a subinterval of C on which neither Y nor Z vanishes. Since $V(d) > 0$, it follows that $U(d)$ is positive. From $U(d) > 0$ follows that either: (A) $Y'(d)Z(d) > 0$, or (B) $Z'(d)Y(d) < 0$.

Consider (A). For the sake of definiteness, we assume $Y'(d) > 0$, $Z(d) > 0$ although a treatment of the case $Y'(d) < 0$, $Z(d) < 0$ merely involves the replacement of Y, Z, Y', and Z' by their negatives in the following proof. It is convenient to treat (A) in three sub-cases: *Case* A_1, $Y(d) > 0$; *Case* A_2, $Y(d) < 0$, $Z'(d) \geqq 0$; *Case* A_3, $Y(d) < 0$, $Z'(d) < 0$.

Case A_1. Since $Z(x)$ and $Y'(d)$ are positive and $Z(x)$ does not vanish on D, we have from (8) (a) that $Y'(x)$ is positive on D. Hence $Y(x)$ is a positive increasing function on D and $Y(x) \geqq Y(d)$ on that interval. From (8) (b), we have

$$\begin{aligned} -Z'(x) &= [-\int_d^\infty r(t)w(t)^2 Y(t)\,dt + H]/w(x)^2 \\ &\geqq Y(d)[-\int_d^\infty rw^2 dt + H]/w^2, \end{aligned}$$

where $H = -w(d)^2 Z'(d)$. Hence

$$-Z(x) = -Z(d) + \int_d^x [-Z'(t)]\,dt \geqq -Z(d) + \int_d^x [Y(d) \int_d^s -r(t)w(t)^2 dt + H]\,ds/w(s)^2.$$

By Lemma II, the integral on the right hand side of this inequality becomes positively infinite with x and hence $\lim_{x\to\infty} Z = -\infty$. From the continuity of $Z(x)$ and $Z(d) > 0$ follows that $Z(x)$ must vanish at least once on D, thereby contradicting our hypothesis that $Z(x) \neq 0$ on D.

Case A_2. From $Z'(d) \geqq 0$, $Y(d) < 0$, $r(x) < 0$, we get that $Y(x)$ is negative on D and hence from (8) (b) we have that $Z'(x) \geqq 0$ on D. Hence $Z(x)$ is a positive non-decreasing function on D and $Z(x) \geqq Z(d) > K > 0$. From equation (8) (a), we have

$$Y'(x) \geqq [K \int_d^x w(t)^2 q(t)\,dt + H]/w(x)^2, \qquad H = w(d)^2 Y'(d),$$

$$Y(x) \geqq Y(d) + \int_d^x [K \int_d^s w(t)^2 q(t)\,dt + H]\,ds/w(s)^2.$$

Applying Lemma II to the integral on the right hand side, we get $\lim_{x\to\infty} Y(x) = +\infty$ and hence $Y(x)$ must vanish at least once on D, since $Y(d) < 0$. This contradicts our hypothesis that $Y(x)$ does not vanish on D.

Case A_3. An examination of equation (8) (b) shows that if for any value, $x = b$, of x on D, we have $Z'(b) \geqq 0$, then for $x > b$, $Z'(x) > 0$ and the argument of *Case* A_3 applies to give a contradiction. Hence $Z'(x) < 0$ on D.

We prove that neither $Y(x)$ nor $Z(x)$ can vanish on $c \leqq x \leqq d$. Assume that one or the other of these functions vanishes on C and let $x = g$ be the last zero of YZ on C. Since $Z'(x)$ is an increasing function of x on G: $g < x < \infty$, we have $Z'(g) < 0$. If $Y(g) = 0$, then $U(g) = Y'(g)Z(g) \geqq 0$ and since $Z(d) > 0$ and $Z(x)$ does not vanish on G, we have $Z(x) > 0$ on G and hence $Y'(g) \geqq 0$. From (8) (a), using $b = g$, we have $Y'(x) > 0$ on G and hence $Y(x) > 0$ on D, thereby contradicting $Y(x) < 0$ on D. If $Z(g) = 0$, then $U(g) = -Z'(g)Y(g) \geqq 0$ or $Z'(g) \geqq 0$, since $Y(g) < 0$. This, however, contradicts $Z'(g) < 0$ and completes the proof of our assertion that neither Y nor Z vanish on C.

From the non-vanishing of Y and Z on C follows that $Z'(c)$ is negative and from $z'(c) = w'(c)Z(c) + w(c)Z'(c) = Z'(c)$ follows that $z'(c)$ is

negative. Let us show that $z'(x)$ is negative on C. Suppose that for a point $x=e$ of C we have $z'(e)=0$. Let $W(x)$ be the solution of the system $W''+p(x)W=0$, $W(e)=1$, $W'(e)=0$, and let $y=Wy^*$, $z=Wz^*$. Since $z'(e)=0$, we have

$$y^{*\prime}(x)=[\int_e^x q(t)W(t)^2z^*(t)\,dt+y^{*\prime}(e)W(e)^2]/W(x)^2$$

(9)

$$z^{*\prime}(x)=[\int_e^x r(t)W(t)^2y^*(t)\,dt]/W(x)^2.$$

Since $y^*<0$ on E: $e\leqq x<\infty$ (y^* has the same sign as y) and since $z^*(e)=z/W>0$, we have that $z^*(x)$ is a positive increasing function on E. Hence $z^*(x)\geqq z^*(e)>K>0$ and

$$y^*(x)\geqq y^*(e)+\int_e^x [K\int_e^s q(t)W(t)^2dt+y'(e)]\,ds/W(s)^2.$$

Applying Lemma II, we get $\lim_{x\to\infty} y^*=+\infty$. But since $y^*(e)=y(e)/W(e)<0$ it follows that y^* and y must vanish on E, thereby contradicting our hypothesis that y does not vanish on D. Hence for every x on C, we have $z'(x)<0$. Furthermore, $y'(x)$ does not vanish on C. For, if $y'(h)=0$, then from $V(h)=-y(h)z'(h)\geqq 0$ follows that $z'(h)$ is positive or zero which contradicts $z'(c)<0$ on C. A similar argument shows that $Y'(x)$ cannot vanish on C. Since $Y'(c)=y'(c)$ and $Y'(d)>0$, it follows that $y'(c)$ is positive or zero on C. Since $y(x)$ is a negative increasing function and $z(x)$ is a positive decreasing function on C, it follows that $R(x)^2=y(x)^2+z(x)^2$ is a decreasing function on C and is consequently less than a positive constant K. Now

$$\theta'(x)=-V(x)/R(x)^2=-V(d)/R(x)^2-[\int_d^x (qz^2-ry^2)\,dt]/R(x)^2$$
$$<-V(d)/K<-H<0,$$

where H is a constant. Hence $-\theta=-\theta(d)-\int_d^x \theta'(t)\,dt>-\theta(d)+H(x-d)$. Hence $\lim_{x\to\infty}\theta=-\infty$ and $y=R\cos\theta$, $z=R\sin\theta$ vanish infinitely often on D. This, however, contradicts the hypothesis that $y(x)$ and $z(x)$ each vanish only a finite number of times on D.

We omit the arguments for (B) since they parallel, step by step, the arguments for (A). In fact, if one wishes to do so he can bring this case under (A) by the use of the transformation $y=-z^*(x)$, $z=y^*$.

THEOREM VIII. *If there exists a positive constant J such that $|q|/(-p)^{1/2} \geqq J$, $|r|/(-p)^{1/2} \geqq J$ on X and if $(q+r)^2 - 4p^2 \leqq 0$ on X, then every solution $y(x)$, $z(x)$ of (3) either oscillates infinitely often on X or else the quantity $R^2 = y^2 + z^2$ decreases steadily approaching zero on some interval G: $g \leqq x < \infty$.*

Proof. Here again we confine our attention to the case $q > 0$ since the same arguments treat the case $q < 0$. If for any value of x, $x = d$, we have $V(d) \geqq 0$, then $V(x) > 0$ on $d < x < \infty$ and Theorem VII applies to prove that the solution oscillates infinitely often on X.

If there is a solution $y(x)$, $z(x)$ of (3) such that $V(x) < 0$ on X, we let d be chosen so that neither y nor z vanishes on D: $d \leqq x < \infty$. If no such d exists, then y and z each vanish infinitely often on X and the theorem is valid. We have

$$(R^2)' = 2(yy' + zz'), \qquad (R^2)'' = 2(yy'' + zz'' + y'^2 + z'^2).$$

If we multiply the equations of (3) by y and z, respectively, and add the resulting equations, we get

$$\tfrac{1}{2}(R^2)'' = y'^2 + z'^2 + Q,$$

where $Q = -py^2 + (q+r)yz - pz^2$ is a positive semi-definite quadratic form. Since $V(x)$ does not vanish on X, y' and z' cannot vanish simultaneously and $(R^2)''$ does not vanish on X. Hence $(R^2)'$ can vanish at most once on X. Choose a point $x = g$ on D so that $(R^2)'$ does not vanish on G: $g \leqq x < \infty$. Either $yz > 0$ on G or $yz < 0$ on that interval and in either case y and z do not vanish on G. An examination of (3) along with the inequalities $p < 0$, $q > 0$, $r < 0$ reveals that either y'' or z'' does not vanish on G. For the sake of definiteness, suppose that y'' does not vanish on G, then y' can vanish at most once on that interval. Let h be chosen greater than g so that y' does not vanish on H: $h \leqq x < \infty$. We show that neither y^2 nor z^2 can be bounded from zero on H. It follows from our hypotheses on $|q|/(-p)^{1/2}$ and $|r|/(-p)^{1/2}$ that $\int_h^x q(t)\,dt$ and $-\int_h^x r(t)\,dt$ become infinite with x and hence if either y^2 or z^2 were bounded from zero, the formula $V(x) = V(h) + \int_h^x (qz^2 - ry^2)\,dt$ would give $\lim_{x\to\infty} V(x) = +\infty$, which contradicts $V(x) < 0$ on X. Since $y(h) \neq 0$, it follows that $yy' < 0$ on H and hence y^2 approaches zero steadily on H. Since z^2 cannot be bounded from zero in a neighborhood of infinity, we can find a point $x = s$ so that $R(s)^2 < R(h)^2$. Since $(R^2)'$ does not vanish on G, it follows that $(R^2)'$

is negative on this interval and since every neighborhood of infinity contains points where R^2 is arbitrarily small, is follows that R^2 approaches zero steadily on G.

COROLLARY. *If we omit the hypothesis* $(q+r)^2-4p^2\leqq 0$ *from Theorem VIII, then we conclude that every solution of* (3) *either oscillates infinitely often on* X *or else* R^2 *takes on values arbitrarily close to zero in every neighborhood of infinity, i. e.,* $\underline{\lim}_{x\to\infty} R^2=0$, *where the underline indicates the lower limit.**

4°. *Case where the independent variable ranges over the entire real axis.* Consider the differential system (3) defined on the entire real axis $X: -\infty<x<+\infty$. Let qr be negative on X and $q(x)$, $r(x)$, and $p'(x)$ be continuous on this interval.

THEOREM IX. *If* $p(x)\equiv 0$ *on* C: $c\geqq x>-\infty$, *and the integrals* $\int_c^{\infty} tq(t)\,dt$, $\int_c^{\infty} tr(t)\,dt$ *become infinite as* x *becomes infinite on* C, *then every solution of* (3) *such that* $V(c)q(c)\leqq 0$ *will oscillate infinitely often on* C.

Proof. Carry the interval C into T: $-c\leqq t<+\infty$ by the change of independent variable $x=-t$. System (3) becomes

$$(10)\qquad d^2Y/dt^2+P(t)Y=Q(t)Z,\qquad d^2Z/dt^2+P(t)Z=S(t)Y,$$

where $P(t)=p(-t)$, $Q(t)=q(-t)$, $S(t)=r(-t)$, $Y(t)=y(-t)$, $Z(t)=z(-t)$. Let $U(t)=Y'(t)Z(t)-Z'(t)Y(t)$, then $V(x)=-U(t)$ and $U(t)=U(-c)+\int_{-c}^{t}[QZ^2-SY^2]\,ds=-V(c)+\int_{-c}^{t}[QZ^2-SY^2]\,ds$. Since $-V(c)q(c)\geqq 0$, it follows that $U(t)$ is a positive increasing function on $-c<t<\infty$ or a negative decreasing function on this interval according as $q(x)$ is positive or negative on C. Also, from the hypotheses on the integrals of $tq(t)$ and $tr(t)$, we have that $\int_{-c}^{t} sQ(s)\,ds$, $\int_{-c}^{t} sS(s)\,ds$ become infinite with t. We may now apply Theorem V to show that the solutions of (10) under investigation oscillate infinitely often on T and from this we immediately conclude that the corresponding solutions of (3) oscillate infinitely often on C.

* The corollary follows when we note that the only use made of the hypothesis $(q+r)^2-4p^2\leqq 0$ in proving Theorem VIII was to show that $(R^2)'$ could not vanish on G.

The same transformation that was used in proving Theorem IX may be used along with Theorem VII to prove

THEOREM X. *If on* C: $c \geqq x > -\infty$, *we have* $p(x) < 0$, $p'(x) \geqq 0$ *and the quantities* $q(x)/[-p(x)]^{\frac{1}{2}}$, $r(x)/[-p(x)]^{\frac{1}{2}}$ *are bounded from zero, then every solution of* (3) *such that* $q(c)V(c) \leqq 0$ *oscillates infinitely often on* C.

If to the hypotheses of Theorems IX and X we add that $\int_c^{\infty} q(t)\,dt$, $\int_c^{\infty} r(t)\,dt$ become infinite as x becomes negatively infinite and $(r+q)^2 - 4p^2 \leqq 0$ on C, respectively, then Theorems VI and VIII are valid on C.

The results of this part of the paper are summarized in the following theorems:

THEOREM XI. *If in system* (3) *we have* p, q, *and* r *continuous on* X: $-\infty < x < +\infty$, $qr < 0$ *on* X, *and either:* (A), $p(x) \equiv 0$ *for* $|x| > c$, *where* c *is any fixed real number, and* $\int_c^{\infty} tq(t)\,dt$, $\int_c^{-\infty} tq(t)\,dt$, $\int_c^{\infty} tr(t)\,dt$, $\int_c^{-\infty} tr(t)\,dt$ *become infinite with* x; *or* (B), $p(x) < 0$ *for* $|x| > c$, $p'(x) \leqq 0$ *for* $x \geqq c$, $p'(x) \geqq 0$ *for* $x \leqq -c$, $p'(x)$ *continuous for* $|x| > c$, *and the quantities* $q(t)/[-p(t)]^{\frac{1}{2}}$, $r(t)/[-p(t)]^{\frac{1}{2}}$ *bounded from zero for* $|x| > c$, *then every solution of* (3) *oscillates infinitely often on* X. *Furthermore, if* $x = a$ *is any point of* X *and* $V(a)q(a) \geqq 0$, *the solution oscillates infinitely often on the positive real axis, while if* $V(a)q(a) \leqq 0$, *it oscillates infinitely often on the negative real axis. In particular, if for any solution* $y(x)$, $z(x)$ *of* (3) *we have* $V(x) = 0$ *at a point of* X, *then the solution oscillates infinitely often on both the positive and negative real axes.*

THEOREM XII. *If in addition to the hypotheses of Theorem XI we have in case* (A) $\int_c^{\infty} q(t)\,dt$, $\int_c^{-\infty} q(t)\,dt$, $\int_c^{\infty} r(t)\,dt$, $\int_c^{-\infty} r(t)\,dt$ *become infinite with* x *and in case* (B) $(q+r)^2 - 4p^2 \leqq 0$, *then every solution of* (3) *such that* $V(a)q(a) \geqq 0$ *oscillates infinitely often in a neighborhood of* $+\infty$ *and either oscillates infinitely often in a neighborhood of* $-\infty$ *or there exists a neighborhood of* $-\infty$ *in which* $R^2 = y^2 + z^2$ *approaches zero steadily and every solution of* (3) *such that* $V(a)q(a) \leqq 0$ *oscillates infinitely often in a neighborhood of* $-\infty$ *and either oscillates infinitely often in a neighborhood of* $+\infty$ *or there exists a neighborhood of* $+\infty$ *in which* R^2 *approaches zero steadily.*

The geometrical characterization of the integral curve of (3) in the yz-plane is one of the following:

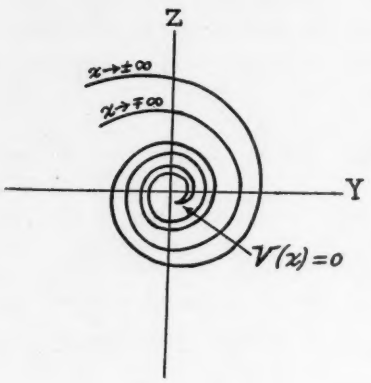

FIGURE I.

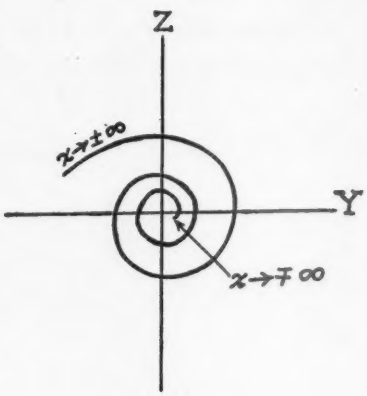

FIGURE II.

III. SYSTEM (3) WITH $q(x)r(x) > 0$.

Consider system (3) where $p(x)$, $q(x)$ and $r(x)$ are of class C' on X: $a \leqq x \leqq b$ and $q(x)r(x) > 0$ on this interval.

Let $w_1(x)$ and $w_2(x)$ be linearly independent solutions of (6) and let $W[w_1, w_2] = w_1(x)w'_2(x) - w_2(x)w'_1(x)$ be the wronskian of these solutions. One easily shows that $W[w_1, w_2]$ is constant on X and is different from zero. If $G(x, t)$ is defined by

$$G(x, t) = [w_1(t)w_2(x) - w_1(x)w_2(t)]/W[w_1, w_2],$$

then the general solution of (3) is given by

$$\begin{aligned} y(x) &= c_1w_1(x) + c_2w_2(x) + \int_a^x G(x,t)q(t)z(t)\,dt \\ z(x) &= c_3w_1(x) + c_4w_2(x) + \int_a^x G(x,t)r(t)y(t)\,dt, \end{aligned} \tag{11}$$

where c_1, c_2, c_3, c_4 are constants.

THEOREM XIII. *If the interval X is such that there exists a non-vanishing solution of (6) on it, then w_1 and w_2 can be so chosen that $W[w_1, w_2] \equiv 1$, $G(x, t) > 0$ on $a \leqq t < x$, and $G(x, t) < 0$ on $x < t \leqq b$.*

Proof. If $w_0(x)$ is the non-vanishing solution of (6) and we let $w_1(x) \equiv |w_0(x)| / |w_0(a)|$ while $w_2(x)$ is the solution of (6) such that

$w_2(a) = 0$, $w'_2(a) = 1$, then $G(x, t)$ and $W[w_1, w_2]$ are readily seen to have the properties that are desired for Theorem XIII.

Let (y_1, z_1), (y_2, z_2), (y_3, z_3), (y_4, z_4) be linearly independent solutions of (3). It follows by well-known considerations that these four solutions form a fundamental system of solutions of (3) and that

$$W = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ z_1 & z_2 & z_3 & z_4 \\ z_1' & z_2' & z_3' & z_4' \end{vmatrix} \equiv \text{a constant} \neq 0.$$

Let u_i and v_i, $(i = 1, 2, 3, 4)$, be the cofactors in W of y_i' and z_i', respectively. Calculation of u_i', v_i', u_i'', v_i'' shows that (u_i, v_i), $(i = 1, 2, 3, 4)$, is a solution of

$$u'' + pu = rv, \qquad v'' + pv = qu. \tag{12}$$

If W^* denotes the determinant obtained from W by changing y_i, y_i', z_i, z_i', respectively, to u_i, u_i', v_i, v_i', $(i = 1, 2, 3, 4)$, then * $W^* \equiv W^3$. W^* is therefore different from zero and (u_i, v_i), $(i = 1, 2, 3, 4)$, form a fundamental system of solutions of (12). If in W^* we let Y_i be the cofactor of u_i' and Z_i the cofactor of v_i', one easily verifies † that $y_i = -WY_i/W^* = -Y_i/W^2$, $z_i = -WZ_i/W^* = -Z_i/W^2$.

Let

$$K_{11}(x, t) = \sum_1^4 y_i(x)u_i(t)/W, \qquad K_{12}(x, t) = \sum_1^4 y_i(x)v_i(t)/W,$$

$$K_{21}(x, t) = \sum_1^4 z_i(x)u_i(t)/W, \qquad K_{22}(x, t) = \sum_1^4 z_i(x)v_i(t)/W.$$

The K's are readily seen to have the following properties:

$$\begin{array}{llll} K_{11}(x,x)=0, & K_{12}(x,x)=0, & K_{21}(x,x)=0, & K_{22}(x,x)=0, \\ (\partial/\partial x)K_{11}(x,x)=1, & (\partial/\partial x)K_{12}(x,x)=0, & (\partial/\partial x)K_{21}(x,x)=0, & (\partial/\partial x)K_{22}(x,x)=1, \\ (\partial/\partial t)K_{11}(x,x)=-1, & (\partial/\partial t)K_{12}(x,x)=0, & (\partial/\partial t)K_{21}(x,x)=0, & (\partial/\partial t)K_{22}(x,x)=-1. \end{array}$$

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* Cf. Bôcher, *Introduction to Higher Algebra*, New York (1919), p. 33.

† This is done by solving the following sets of algebraic equations:

$$\sum_1^4 u_iY_i = 0, \quad \sum_1^4 u_i'Y_i = W^*, \quad \sum_1^4 v_iY_i = 0, \quad \sum_1^4 v_i'Y_i = 0,$$

$$\sum_1^4 u_iy_i = 0, \quad \sum_1^4 u_i'y_i = -W, \quad \sum_1^4 v_iy_i = 0, \quad \sum_1^4 v_i'y_i = 0,$$

and the corresponding equations for z_i and Z_i.

For each fixed x on X, (K_{11}, K_{12}) and (K_{21}, K_{22}) are solutions of (12). For each fixed t on X, (K_{11}, K_{21}) and (K_{12}, K_{22}) are solutions of (3).

THEOREM XIV. *The functions K_{11}, K_{12}, K_{21}, K_{22} are uniquely determined for system* (3), *i. e., they are independent of the particular fundamental system of solutions used in their construction.*

Proof. Let $K^*_{ij}(x, t)$, $(i, j = 1, 2)$, be formed for a different fundamental system of solutions of (3) and suppose that $K^*(c, d) \neq K(c, d)$ for some values of i and j and for some c and d such that $a \leqq c$, $d \leqq b$. $K_{1j}(x, d)$, $K_{2j}(x, d)$ and $K^*_{1j}(x, d)$, $K^*_{2j}(x, d)$ are solutions of system (3) and both have the same initial values at $x = d$. Hence by the uniqueness part of the fundamental existence theorem, $K_{ij}(x, d) = K^*_{ij}(x, d)$ on $a \leqq x \leqq b$, thereby contradicting $K_{ij}(c, d) \neq K^*_{ij}(c, d)$.

THEOREM XV. *The functions K_{ij} have the following properties:*

$$K_{11}(x, t) = -K_{22}(t, x), \quad K_{21}(x, t) = -K_{21}(t, x),$$
$$K_{12}(x, t) = -K_{12}(t, x), \quad K_{22}(x, t) = -K_{11}(t, x).$$

Proof. Following Theorem XIV we can and will form the K's from a fundamental system of solutions that has the property of making $W \equiv 1$. $y_i = v_i$, $z_i = u_i$ is a solution of (3), since u_i, v_i is a solution of (12) and $W = W^* = 1$. Hence the cofactors of v_i' and u_i' in W are respectively $-z_i$ and $-y_i$. Substituting we obtain

$$K_{11}(x, t) = \sum_1^4 y_i(x)u_i(t) = -\sum_1^4 v_i(x)z_i(t) = -K_{22}(t, x),$$

$$K_{12}(x, t) = \sum_1^4 y_i(x)v_i(t) = -\sum_1^4 v_i(x)y_i(t) = -K_{12}(t, x),$$

$$K_{21}(x, t) = \sum_1^4 z_i(x)u_i(t) = -\sum_1^4 u_i(x)z_i(t) = -K_{21}(t, x),$$

$$K_{22}(x, t) = \sum_1^4 z_i(x)v_i(t) = -\sum_1^4 u_i(x)y_i(t) = -K_{11}(t, x).$$

Theorem XV is important in that it gives the values of the K's at every point of $x < t \leqq b$ in terms of the values of these functions on $a \leqq t < x$.

The general solution of

$$(13) \qquad y'' + py - qz = h_1(x)z, \qquad z'' + pz - ry = h_2(x)y,$$

where h_1 and h_2 are continuous real functions of class C' on X and $(q + h_1)(r + h_2) \neq 0$ on X, is given by the pair of integral equations

$$(14)\qquad \begin{aligned} y(x) &= \sum_1^4 c_i y_i(x) + \int_a^x [K_{11}(x,t)h_1(t)z(t) + K_{12}(x,t)h_2(t)y(t)]dt \\ z(x) &= \sum_1^4 c_i z_i(x) + \int_a^x [K_{21}(x,t)h_1(t)z(t) + K_{22}(x,t)h_2(t)y(t)]dt, \end{aligned}$$

where c_1, c_2, c_3, c_4 are constants. The equivalence of systems (13) and (14) may be verified by substitution.

Definition. The interval X is said to be regular of the first kind for* (3) *if $K_{ij}(x, t) > 0$, $(i, j = 1, 2)$, for each x on X^*: $a < x \leqq b$ and every t on $a \leqq t < x$.*

Definition. The interval X is regular of the second kind for (3) *if $K_{11} > 0$, $K_{22} > 0$, $K_{12} < 0$, $K_{21} < 0$, for each x on X^* and for every t on $a \leqq t < x$.*

THEOREM XVI. *If $r > 0$, $q > 0$, on X and there exists a solution of* (6) *that does not vanish on X, then X is regular of the first kind for* (3).

Proof. Let $G(x, t)$ be defined as indicated under Theorem XIII and assume that for some value of x, $x = c$, on X^* and for some t, $t = d$, on $a \leqq t < c$, at least one K_{ij} is less than or equal to zero. Since $K_{11}(c, t)$, $K_{12}(c, t)$ is a solution of (12), we have

$$(15)\qquad \begin{aligned} &\text{(a)}\quad K_{11}(c,t) = -w_2(t) + \int_a^t G(t,s)r(s)K_{12}(c,s)ds \\ &\text{(b)}\quad K_{12}(c,t) = \int_a^t G(t,s)q(s)K_{11}(c,s)ds. \end{aligned}$$

Since $w_2(t) < 0$ for $t < c$ and $(\partial/\partial t)K_{11}(c, c) = -1$, $K_{11}(c, c) = 0$, it follows that for a sufficiently small positive ϵ, $K_{11}(c, t) > 0$ on $c - \epsilon \leqq t < c$. An examination of equation (15) (b) shows that the integrand is negative on the interval $c - \epsilon \leqq t < c$ for $s > t$ and since the upper limit of integration is smaller than the lower, it follows that $K_{12}(c, t)$ is positive on this interval. If either $K_{11}(c, t)$ or $K_{12}(c, t)$ vanishes at $t = d$, we let $t = e$ be the greatest zero of the product of these two functions on $d \leqq t < c$. If $K_{11}(c, e) = 0$, then equation (15) (a) yields

$$(16)\qquad 0 = -w_2(e) + \int_a^e G(e,s)r(s)K_{12}(c,s)ds.$$

Both terms on the right hand side of equation (16) are positive and hence this equation is impossible. Similar reasoning when applied to equation (15) (b) yields a contradiction when $K_{12}(c, e) = 0$ is assumed.

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* Cf. Birkhoff, *loc. cit.*, page 113, and Reynolds, *loc. cit.*, page 223.

Since $K_{21}(c,t)$, $K_{22}(c,t)$ is a solution of (12), we have

$$(17)\qquad \begin{aligned} K_{21}(c,t) &= \int_c^t G(t,s)r(s)K_{22}(c,s)\,ds \\ K_{22}(c,t) &= -w_2(t) + \int_c^t G(t,s)q(s)K_{21}(c,s)\,ds \end{aligned}$$

and an application of reasoning that is entirely analogous to that used in treating K_{11} and K_{12} shows that neither $K_{21}(c,d)$ nor $K_{22}(c,d)$ can be zero. Hence none of the K's can vanish at (c,d) and it follows from this contradiction that Theorem XVI must hold.

Obvious modifications of the proof of Theorem XVI establish:

THEOREM XVII. *If $r<0$, $q<0$ on X and if there exists a solution of* (6) *that does not vanish on X, then X is regular of the second kind for* (3).

We note as an obvious consequence of the foregoing work that any regular interval for (3) is also regular and of the same kind for (12) and conversely, any regular interval for (12) is also regular and of the same kind for (3).

In the presence of the requirement $rq>0$ it is true that every finite interval X can be broken into a finite number of regular intervals for (3) all of which are of the first kind or else they all are of the second kind. One merely notes that a solution of (6) has only a finite number of zeros on any interval of finite length. In particular, if $p(x)<0$ equation (6) has solutions that do not vanish on any sub-interval of the real axis and in this case the entire real axis is regular of one kind for (3) and (12).

THEOREM XVIII. *If X is a regular interval of the first (second) kind for system* (3) *and (Y,Z) and (y,z), respectively, are non-identically vanishing solutions of* (3) *and* (13) *such that $y(c)=Y(c)$, $y'(c)=Y'(c)$, $z(c)=Z(c)$, $z'(c)=Z'(c)$, $a\leqq c\leqq b$, and if $h_1(x)\equiv 0$ on X, then the first zero* of $y(x)$ on $c<x\leqq b$ follows (precedes) or precedes (follows)*

* The proof that the zeros of y, z, Y, and Z on X cannot have a finite limit point is as follows:

Assume that $x=c$ is a limit point of the zeros of one of these quantities, for example $y(x)$. Rolle's theorem and continuity considerations show that $x=c$ is also a limit point of zeros of $y'(x)$, $y''(x)$, and $y'''(x)$ and furthermore, $y(c)=y'(c)=y''(c)=y'''(c)=0$. Now $z(c)=[y''(c)+p(c)y(c)]/q(c)=0$, $z'(c)=[y'''(c)+p'(c)y(c)+p(c)y'(c)-q'(c)z(c)]/q(c)=0$. Hence $y(c)=y'(c)=z(c)=z'(c)=0$ and by the fundamental existence theorem, $y(x)\equiv z(x)\equiv 0$ on X. This contradicts $y(x)$ and $z(x)$ not identically zero on X. Similar arguments when applied to the proper equations of (3) or (13) treat the cases $z(x)$, $Y(x)$, $Z(x)$.

the first zero of $Y(x)$ on this interval according as $h_2(x) \geqq 0$ on $c < x \leqq b$ or $h_2(x) \leqq 0$ on this interval, in any case the equality sign holding on at most a null set of points.

Proof. Let a in formula (14) be replaced by c and note that $\sum_1^4 c_i y_i(x)$, $\sum_1^4 c_i z_i(x)$ is a solution of (3) that has the same initial conditions at $x = c$ as $Y(x)$, $Z(c)$. Using the uniqueness part of the fundamental existence theorem for (3), system (14) becomes

$$\text{(18)} \quad \begin{aligned} &\text{(a)} \quad y(x) = Y(x) + \int_c^x K_{12}(x,t) h_2(t) y(t)\, dt \\ &\text{(b)} \quad z(x) = Z(x) + \int_c^x K_{22}(x,t) h_2(t) y(t)\, dt. \end{aligned}$$

It follows readily by differentiation that the lowest derivatives of y and Y that do not vanish at $x = c$ are equal at this point. Hence if $x = d$ is a point such that $d > c$ and neither y nor Y vanish on $c < x \leqq d$, then on this interval $Y(x)y(x)$ is positive.

If $h_2(x) > 0$ almost everywhere on X and X is regular of the first kind, we have that $\int_c^x K_{12}(x) h_2(t) y(t)\, dt$ is different from zero and has the same sign as $Y(x)$ for every x on $c < x \leqq b$ that is less than or equal to the smallest value of x on this interval for which yY vanishes. Hence the product yY must vanish for a smaller value of x on $c < x \leqq b$ than the smallest value on this interval for which $y(x)$ vanishes and our theorem follows for this case.

If X is regular of the second kind, we transpose the integral term of (18) (a) to the other side and repeat the above argument with respect to $Y(x)$ to get the desired conclusion. If $h_2(x) \leqq 0$ on X, a repetition of the foregoing argument with obvious modifications treats all of the cases that arise here.

If $h_2(x) \equiv 0$ on X and (y, z), (Y, Z) are defined as in Theorem XVIII, equations (14) become

$$\text{(19)} \quad \begin{aligned} y(x) &= Y(x) + \int_c^x K_{11}(x,t) h_1(t) z(t)\, dt \\ z(x) &= Z(x) + \int_c^x K_{21}(x,t) h_1(t) z(t)\, dt \end{aligned}$$

and reasoning entirely analogous to that used in proving Theorem XVIII establishes

THEOREM XIX. *If X is a regular interval of the first (second) kind for (3) and $h_2(x) \equiv 0$ on X, then the first zero of $z(x)$ on $c < x \leqq b$ follows (precedes) or precedes (follows) the first zero of $Z(x)$ on that interval according as as $h_1(x) > 0$ almost everywhere on $c < x \leqq b$ or $h_1(x) < 0$ almost everywhere on that interval.*

Using the properties of $K_{ij}(x, t)$, $(i, j = 1, 2)$, that were established in Theorems XIV and XV, we establish the following theorems on $a \leqq x < c$:

THEOREM XX. *Under the hypotheses of Theorem XIX, if X is a regular interval of the first (second) kind for (3), the last zero of $y(x)$ on $a \leqq x < c$ precedes (follows) or follows (precedes) the last zero of $Y(x)$ on this interval according as $h_2(x) > 0$ almost everywhere on $a \leqq x < c$ or $h_2(x) < 0$ almost everywhere on this interval.*

THEOREM XXI. *Under the hypotheses of Theorem XIX, if X is a regular interval of the first (second) kind for (3), the last zero of $z(x)$ on $a \leqq x < c$ precedes (follows) or follows (precedes) the last zero of $Z(x)$ on that interval according as $h_1(x) > 0$ almost everywhere on $a \leqq x < c$ or $h_1(x) < 0$ almost everywhere on that interval.*

THEOREM XXII. *Let (y, z) and (Y, Z) be defined as in Theorem XVIII and $q(c)[q(c) + h_1(c)] > 0$, $r(c)[r(c) + h_2(c)] > 0$. If X is regular of the first (second) kind for (3) and the first non-vanishing derivative of $y(x)$ at $x = c$ has the same (opposite) sign as the first derivative of $z(x)$ that does not vanish at $x = c$, then the first zero of yz on $c < x \leqq b$ follows (precedes) or precedes (follows) the first zero of YZ on this interval according as $h_1(x) > 0$, $h_2(x) > 0$ almost everywhere on $c < x \leqq b$ or $h_1(x) < 0$, $h_2(x) < 0$ almost everywhere on that interval.*

Proof. Following the reasoning used under Theorem XVIII we may write the lower limits of integrations in system (14) as c and replace $\sum_1^4 c_i y_i(x)$ by $Y(x)$ and $\sum_1^4 c_i z_i(x)$ by $Z(x)$ in this system. Calculation of the derivatives of Y, Z, y, and z together with the hypotheses on $q(c)[q(c) + h_1(c)]$ and $r(c)[r(c) + h_2(c)]$ shows that if $x = d$ is any point such that $yzYZ \neq 0$ on $c < x \leqq d$, then $y(d)Y(d) > 0$, $z(d)Z(d) > 0$. An examination of the right hand side of (14) shows that if X is regular of the first kind, and the lowest non-vanishing derivatives of $y(x)$ and $z(x)$ at $x = c$ are of the same sign, the integrals will have the same signs (opposite

signs) as $Y(x)$ and $Z(x)$ if $h_1 > 0$, $h_2 > 0$ ($h_1 < 0$, $h_2 < 0$) almost everywhere on X. Application of an argument of the type of that was used in proving Theorems XVIII and XIX shows that the first zero of yz on $c < x \leqq b$ must precede the first zero of YZ on this interval.

An entirely similar argument demonstrates the theorem for the cases where X is regular of the second kind.

An argument that follows exactly the same lines as that used in proving Theorem XXII establishes the following theorem:

THEOREM XIII. *Let $q(c)[q(c) + h_1(c)]$ and $r(c)[r(c) + h_2(c)]$ be positive and let y, z, Y, Z, be defined as in Theorem* XVIII. *Let $x = e$ be the first zero of yz on $c < x \leqq b$ and let $x = f$ be the first zero of YZ on this interval. Let X be regular of the first (second) kind for* (3) *and let the lowest non-vanishing derivatives of y and z at $x = c$ have opposite (the same) signs. Then if* $\left\{\begin{matrix} z(e) = 0 \\ y(e) = 0 \end{matrix}\right\}$, *we have* $\left\{\begin{matrix} f < e\ (f > e) \\ f > e\ (f < e) \end{matrix}\right\}$ *or* $\left\{\begin{matrix} f > e\ (f < e) \\ f < e\ (f > e) \end{matrix}\right\}$ *according as $h_1 > 0$, $h_2 < 0$ almost everywhere on $c < x \leqq b$ or $h_1 < 0$, $h_2 > 0$ on that interval.*

It is evident that Theorems XIV and XV may be used to establish theorems on $a \leqq x < c$ that are the exact analogues of Theorems XXII and XXIII.

Definition. By the $\left\{\begin{matrix} \textit{forward} \\ \textit{backward} \end{matrix}\right\}$ *interval of oscillation* of* (3) *at $x = c$ we mean the subinterval* $\left\{\begin{matrix} c \leqq x \leqq d \\ e \leqq x \leqq c \end{matrix}\right\}$ *of X, if such exists, such that* $\left\{\begin{matrix} d \\ e \end{matrix}\right\}$ *is the* $\left\{\begin{matrix} \textit{smallest} \\ \textit{largest} \end{matrix}\right\}$ *number having the property that every solution of* (3) *whose y or z vanishes at $x = c$ also has a zero of $y(x)z(x)$ on* $\left\{\begin{matrix} c < x \leqq d \\ e \leqq x < c \end{matrix}\right\}$.

THEOREM XXIV. *If $q(x)r(x) > 0$ on X, the backward and forward intervals of oscillation for* (3), *if they exist, are greater in length than the corresponding intervals† for* (6).

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* Cf. Birkhoff, *loc. cit.*, page 119.

† These intervals for (6) are determined by the zeros that immediately precede and follow $x = c$ of a solution of (6) that vanishes at $x = a$.

Proof. Let $w(x)$ and $W(x)$ be the solutions of (6) having the values $w(c) = 0$, $w'(c) = 1$, $W(c) = 1$, $W'(c) = 0$, where $x = c$ is any point of X, and form $G(x, t)$ from these solutions. $y(x) = \int_c^x G(x, t) q(t) z(t)\, dt + w(x)$, $z(x) = \int_c^x G(x, t) r(t) y(t)\, dt$ represents the solution of (3) such that $y(c) = z(c) = z'(c) = 0$, $y'(c) = 1$. If $x = d$ is the first zero of $w(x)$ such that $d > c$, then $G(d, t)$ is positive on $c < x < d$ and vanishes at $x = c$ and $x = d$. An examination of the above equations for $y(x)$ and $z(x)$ shows that neither y nor z can vanish on $c < x \leqq d$. Similar considerations treat the backward intervals of oscillation.

COROLLARY. *If $qr > 0$ and (6) has a non-oscillatory solution on any interval X, there exist solutions of (3) such that either y or z does not vanish on X.*

IV. APPLICATION TO EQUATIONS (1) AND (2).

We indicate briefly how the foregoing treatment applies to the general fourth order self-adjoint equation (1) and to the second order equation (2).

For equation (1) we have $p(x) = L(x)/2K(x)$, $q(x) = 1/K(x)$, $r(x) = L''(x)/2 + L(x)^2/4K(x) - G(x)$. The conditions that we have imposed on p, q, and r in the various theorems are conditions on the above combinations of the coefficients of (1). The conclusions of these theorems apply to $u(x)$, the solution of (1), and the combination $Ku'' + Lu/2$. In the special but important case $L(x) = 0$, we have $p(x) = 0$, $r(x) = -G(x)$ and the conditions of the theorems become very simple.

Equation (2) is equivalent to system (3) where $q(x) = -r(x)$. This property simplifies several of the hypotheses of the theorems of the paper. In particular, the condition $(q + r)^2 - 4p^2 \leqq 0$ is automatically satisfied. Let $u(x) = u_1(x) + iu_2(x)$ in equation (2), then

$$
\begin{aligned}
u_1 \exp\left[\tfrac{1}{2}\int_a^x p_1\, dt\right] &= y \cos \int_a^x p_2\, dt/2 + z \sin \int_a^x p_2\, dt/2 \\
u_2 \exp\left[\tfrac{1}{2}\int_a^x p_1\, dt\right] &= -y \sin \int_a^x p_2\, dt/2 + z \cos \int_a^x p_2\, dt/2
\end{aligned} \tag{20}
$$

and the determinant of coefficients of y and z is identically unity.

The following theorem concerning equations of type (20) forms a connecting link between the properties of y and z and those of u_1 and u_2.

THEOREM XXV. *If $y(x)$, $z(x)$, $a(x)$, and $b(x)$ are of class C' on X and $(a'b - ab')(y'z - yz') \geqq 0$, $V(x) = y'z - z'y \neq 0$, $a \neq 0$, $b \neq 0$, on X, then*

1). *The zeros of $y(x)$ and $z(x)$ separate each other on X.*

2). *If $u = ay - bz$, $v = by + az$, there is at least one zero each of u and v between each pair of zeros of either y or z on X.*

3). *The zeros of u and v separate each other on X.*

4). *There is only one zero each of u and v between each pair of consecutive zeros of y and between each pair of consecutive zeros of z.*

Proof. 1). Let C and D denote two zeros of y on X and assume that z does not vanish between C and D. $z(x)$ cannot vanish at either C or D since this would cause V to vanish at such a point. Consider the continuous function y/z on the closed interval bounded by C and D. The derivative $[y/z]' = V(x)/z^2$ does not vanish on this interval and hence by Rolle's theorem y/z cannot vanish at both C and D. This contradiction shows that z vanishes at least once between each pair of zeros of y. Similar considerations of z/y shows that y must vanish at least once between each pair of zeros of z.

2). $y(x)$ and $z(x)$ cannot have a finite limit point for their zeros on X, for continuity considerations together with conclusion 1) above would cause both $y(x)$ and $z(x)$ to vanish at such a point and hence $V(x)$ would necessarily vanish there. Let $x = c$ and $x = d$ be consecutive zeros of y on X and assume that $v(x)$ does not vanish on $c < x < d$. An examination of the equations connecting y, z, u, and v, together with the restriction on $V(x)$, shows that no two of these quantities can vanish simultaneously on X. Since $z(x)$ vanishes exactly once on $c < x < d$, we have from $a(x) \neq 0$, $v(c) = a(c)z(c)$, $v(d) = a(d)z(d)$, that $v(c)v(d) < 0$ and hence $v(x)$ must vanish at least once on $c < x < d$. Similar arguments treat all of the other cases.

3). Since u and v do not vanish simultaneously, we establish 3) by considerations similar to those used in proving 1). We note that $(u/v)' = [(a'b - ab')(y^2 + z^2) + (a^2 + b^2)(y'z - yz')]/v^2$, when $v(x) \neq 0$, and this quantity does not vanish on X.

4). In the presence of conclusion 3), we may use the same argument that was used under 2) and show that u and v cannot have finite limit points for their zeros on X. Hence we may order the zeros of these functions and speak of the consecutive zeros of $u(x)$ or $v(x)$ on X. Solving for $y(x)$

and $z(x)$, we obtain $y(x)=(au+bv)/(a^2+b^2)$, $z(x)=(av-bu)/(a^2+b^2)$. The argument that was used under 2) to show that v vanishes at least once between each pair of consecutive zeros of $y(x)$ can now be applied to show that $y(x)$ and $z(x)$ each vanish at least once between each pair of consecutive zeros of $u(x)$ and between each pair of consecutive zeros of $v(x)$. This together with 2) establishes part 4).

In conclusion, it is worthwhile to point out that the foregoing treatment does not touch the cases where $q(x)r(x)$ changes signs on the interval under consideration. The methods of the present paper do not seem to lend themselves readily to a treatment of these cases except in so far as it is possible to break up the given interval into sub-intervals upon each of which either $q(x)r(x)\leqq 0$ or $q(x)r(x)\geqq 0$. Other important cases omitted from the present paper are those that arise when $p(x)$ neither vanishes identically nor remains negative on interval considered. All of these cases are important and it would be of interest to develop the properties of the solutions of such systems.

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